Problem 1: Suppose $A$, $B$, $C$ and $D$ are arbitrary sets. **Prove or disprove:** \((A \times C) \cap (B \times D) = (A \cup B) \times (C \cup D)\).

**Solution:** The statement is false. The following is a counterexample.

\[
A = \{1\}, \quad B = \emptyset, \quad C = \{2\} \quad \text{and} \quad D = \emptyset.
\]

Here, \(A \times C = \{(1, 2)\}\) and \(B \times D = \emptyset\). Thus, \((A \times C) \cap (B \times D) = \emptyset\).

However, \(A \cup B = \{1\}\) and \(C \cup D = \{2\}\). Hence, \((A \cup B) \times (C \cup D) = \{(1, 2)\}\).

Hence, \((A \times C) \cap (B \times D) \neq (A \cup B) \times (C \cup D)\).

Problem 2: Suppose $A$, $B$ and $C$ are nonempty finite sets with cardinalities \(n_a\), \(n_b\) and \(n_c\) respectively. Derive a condition on \(n_a\), \(n_b\) and \(n_c\) so that there is a one-to-one function from \(A \times B\) to \(C\). Assuming that condition, give an expression for the number of one-to-one functions from \(A \times B\) to \(C\).

**Solution:** For finite sets $X$ and $Y$, the condition for the existence of a one-to-one function from $X$ to $Y$ is \(|Y| \geq |X|\). When this condition holds, the number of one-to-one function from $X$ to $Y$ is given by \(|Y|!/(|Y| - |X|)!\).

Here, \(X = A \times B\). So, \(|X| = |A \times B| = n_a \times n_b\). Further, \(Y = C\). So, \(|Y| = |C| = n_c\).

Therefore,

(i) The condition for the existence of a one-to-one function from \(A \times B\) to \(C\) is: \(n_c \geq n_a \times n_b\).

(ii) Assuming that the condition given in (i) holds, the number of one-to-one functions from \(A \times B\) to \(C\) is: \((n_c)!/[n_c - (n_a \times n_b)]!\).

Problem 3: Let \(P = \{1, 2, 3, 4\}\) and \(Q = \{u, v, w, x, y\}\). How many functions from \(P\) to \(Q\) are neither one-to-one nor map 3 to \(y\)? Show work.

**Solution:** The total number of functions from \(P\) to \(Q\) is \(5^4 = 625\). Suppose \(N\) is the number of functions from \(P\) to \(Q\) which are either one-to-one or map the element 3 to \(y\). Then, the answer to the problem is \(625 - N\).

We can compute the value of \(N\) as follows. Let \(S_1\) denote the set of one-to-one functions from \(P\) to \(Q\) and let \(S_2\) denote the set of functions from \(P\) to \(Q\) that map the element 3 to \(y\). Thus, \(N = |S_1 \cup S_2|\). From the inclusion-exclusion formula, we have

\[
N = |S_1 \cup S_2| = |S_1| + |S_2| + |S_1 \cap S_2|.
\]

We now show how each cardinality value on the right side of the above equation can be computed.

(a) \(|S_1|\) is the number of of one-to-one functions from \(P\) to \(Q\). In constructing a one-to-one functions from \(P\) to \(Q\), the number of choices for the elements 1, 2, 3 and 4 are 5, 4, 3 and 2 respectively. Thus, \(|S_1| = 5 \times 4 \times 3 \times 2 = 120\).
Problem 5:
Let \( |S_2| \) is the number of functions from \( P \) to \( Q \) that map the element 3 to \( y \). To compute \( |S_2| \), note that there are 5 choices for each of the elements 1, 2 and 4 while there is only one choice for the element 3. Thus, \( |S_2| = 5 \times 5 \times 5 \times 1 = 125 \).

(c) To compute \( |S_1 \cap S_2| \), which is the number of functions from \( P \) to \( Q \) which are one-to-one and map the element 3 to \( y \), we reason as follows. Any such function is a one-to-one function from the set \( \{1,2,4\} \) to the set \( \{u,v,w,x\} \). In constructing such a function, the number of choices for the elements 1, 2 and 4 are respectively 4, 3 and 2. Thus, \( |S_1 \cap S_2| = 4 \times 3 \times 2 = 24 \).

Now, using the inclusion-exclusion formula, we have \( N = 120 + 125 - 24 \) or \( N = 221 \).

So, the required answer is \( 625 - 221 = 404 \).

Problem 4: Suppose \( p, q \) and \( r \) are propositions such that \( (\neg p \land \neg q) \rightarrow \neg r, \neg q \rightarrow \neg p \) and \( \neg r \rightarrow p \) are all true. Prove that \( q \) must be true. No credit will be given if you use a truth table to prove this result.

Solution: We will use a proof by contradiction. So, suppose \( q \) is false; that is, \( \neg q \) is true.

(1) Since \( \neg q \) is true (from our assumption) and \( \neg q \rightarrow \neg p \) is true (given), \( \neg p \) is true (by modus ponens); that is, \( p \) is false.

(2) Since \( \neg p \) is true (from (1)) and \( \neg q \) is true (from our assumption), \( \neg p \land \neg q \) is true (property of and operator).

(3) Since \( \neg p \land \neg q \) is true (from (2)) and \( (\neg p \land \neg q) \rightarrow \neg r \) is true (given), \( \neg r \) is true (by modus ponens).

(4) Since \( \neg r \) is true (by (3)) and \( \neg r \rightarrow p \) is true (given), \( p \) is true (by modus ponens). However, this contradicts the conclusion in (1) that \( p \) is false.

Thus, \( q \) must be true.

Problem 5: Let \( U = \{1,2,3,4\} \). Assume that variables \( x \) and \( y \) take on values from \( U \). Let \( P(x,y) \) denote the predicate “\( x^2 < y + 1 \)” and let \( Q(x,y) \) denote the predicate “\( x^2 + y^2 < 12 \)”.

Give the truth values of each of the following propositions and explain how you arrived at your result:

(i) \((\forall x)(\forall y) \ P(x,y) \)

(ii) \((\exists x)(\forall y) \ P(x,y) \)

(iii) \((\exists x)(\exists y) \ Q(x,y) \)

(iv) \((\forall x)(\exists y) \ Q(x,y) \)

Solution: For this problem, the universe for the variables \( x \) and \( y \) is \( \{1,2,3,4\} \).

(i) The proposition \((\forall x)(\forall y) \ P(x,y) \) is false.

Explanation: When \( x = 2 \) and \( y = 1 \), we have \( x^2 = 4 \) and \( y + 1 = 2 \). So, \( x^2 < y + 1 \) is false for this combination of \( x \) and \( y \) values. In other words, \( P(2,1) \) is false. Therefore, \((\forall x)(\forall y) \ P(x,y) \) is false.

(ii) The proposition \((\exists x)(\forall y) \ P(x,y) \) is true.

Explanation: Choose \( x = 1 \). Now, for each value \( y \in \{1,2,3,4\} \), notice that \( x^2 = 1 \) is less than \( y + 1 \). Therefore, \((\exists x)(\forall y) \ P(x,y) \) is true.

(iii) The proposition \((\exists x)(\exists y) \ Q(x,y) \) is true.

Explanation: Let \( x = y = 1 \). Then \( x^2 + y^2 = 2 < 12 \). Thus \( Q(1,1) \) is true. Therefore, \((\exists x)(\exists y) \ Q(x,y) \) is true.
(iv) The proposition \((\forall x)(\exists y) \, Q(x, y)\) is false.

**Explanation:** Let \(x = 4\). Thus, \(x^2 = 16\). For this value of \(x\), no value for \(y \in \{1, 2, 3, 4\}\) can ensure that \(x^2 + y^2 < 12\). Thus, \((\forall x)(\exists y) \, Q(x, y)\) is false.

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**Problem 6:** Let \(T\) be the set of all \(2 \times 2\) matrices with integer entries. Consider the function \(f : T \times T \rightarrow T\) defined by \(f(A, B) = A + B\), where ‘+’ denotes the sum of the two matrices.

(a) Is \(f\) a one-to-one function? Justify your answer.

(b) If \(f\) an onto function? Justify your answer.

**Solution:**

**Part (a):** \(f\) is not one-to-one. To see this, let

\[
A_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \quad \text{and} \quad B_2 = \begin{bmatrix} -2 & -2 \\ -1 & -2 \end{bmatrix}.
\]

Now,

\[
f(A_1, B_1) = A_1 + B_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad f(A_2, B_2) = A_2 + B_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.
\]

Thus, \(f\) maps two different inputs, namely \((A_1, B_1)\) and \((A_2, B_2)\), to the same result (namely, the \(2 \times 2\) matrix consisting of all zeros). So, \(f\) is not one-to-one.

**Part (b):** \(f\) is onto.

To prove this statement, we must show that for any \(2 \times 2\) integer matrix \(A\), there are \(2 \times 2\) integer matrices \(X\) and \(Y\) such that \(f(X, Y) = A\); that is, \(X + Y = A\). This can be achieved by choosing \(X\) as \(A\) itself and \(Y\) as the \(2 \times 2\) integer matrix in which all four entries are zero. Therefore, \(f\) is onto.

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**Problem 7:** Let \(\mathcal{R}^+\) denote the set of positive real numbers. Consider the binary relation \(B\) on \(\mathcal{R}^+\) defined as follows: \((x, y) \in B\) if and only if \([x] = [y]\). Is \(B\) symmetric? antisymmetric? Explain your answer in each case.

**Solution:**

**Part (a):** \(B\) is not symmetric.

To see this, notice that \((7.3, 6.7)\) is in \(B\) since \([7.3] = [6.7] = 7\). However, \((6.7, 7.3)\) is not in \(B\) since \([6.3] = 6\) and \([7.3] = 8\).

**Part (b):** \(B\) is antisymmetric. We will prove this result by showing that if \((x, y) \in B\) and \((y, x) \in B\), then \(x = y\).

Let \((x, y) \in B\) and \((y, x) \in B\). By the definition of \(B\), we have

\[
[x] = [y] \quad (1)
\]

\[
[y] = [x] \quad (2)
\]

Since \(x\) and \(y\) are positive, we know that

\[
[x] \leq [x] \quad (3)
\]

\[
[y] \leq [y]. \quad (4)
\]

Now,
Thus, $[y] \leq [y]$. Combining this with the inequality $[y] \leq [y]$ (Equation (4)), we conclude that $[y] = [y]$. Thus, $y$ must be an integer and $[y] = [y] = y$. In a similar manner, $[x] = [x] = x$. Since $x$ and $y$ are integers, the equation $[x] = [y]$ above immediately implies that $x = y$. Thus, the relation $B$ is antisymmetric.

Problem 8: Suppose $R_1$ and $R_2$ are equivalence relations over a set $A$. Prove that $R_1 \cap R_2$ is also an equivalence relation.

Proof: We must show that $R_1 \cap R_2$ is reflexive, symmetric and transitive using the fact that both $R_1$ and $R_2$ satisfy those properties.

(a) Proof that $R_1 \cap R_2$ is reflexive: Consider any $x \in A$. We must show that $(x, x) \in R_1 \cap R_2$. This can be done as follows.

Since $R_1$ and $R_2$ are reflexive, $(x, x) \in R_1$ and $(x, x) \in R_2$. Therefore, $(x, x) \in R_1 \cap R_2$.

(b) Proof that $R_1 \cap R_2$ is symmetric: Let $(x, y) \in R_1 \cap R_2$. We must show that $(y, x) \in R_1 \cap R_2$. This can be done as follows.

Since $(x, y) \in R_1 \cap R_2$, $(x, y) \in R_1$ and $(x, y) \in R_2$. Since $R_1$ and $R_2$ are symmetric, $(y, x) \in R_1$ and $(y, x) \in R_2$. Therefore, $(y, x) \in R_1 \cap R_2$.

(c) Proof that $R_1 \cap R_2$ is transitive: Let $(x, y) \in R_1 \cap R_2$ and $(y, z) \in R_1 \cap R_2$. We must show that $(x, z) \in R_1 \cap R_2$. This can be done as follows.

Since $(x, y) \in R_1 \cap R_2$, $(x, y) \in R_1$ and $(x, y) \in R_2$. Likewise, $(y, z) \in R_1$ and $(y, z) \in R_2$. Since $R_1$ and $R_2$ are transitive, $(x, z) \in R_1$ and $(x, z) \in R_2$. Therefore, $(x, z) \in R_1 \cap R_2$.

This completes the proof.

Problem 9: How many solutions are there to the equation $x_1 + x_2 + x_3 + x_4 = 31$, if $x_1$, $x_2$, $x_3$ are non-negative integers and $x_4$ is a positive multiple of 8? (You may leave the answer as an expression consisting of binomial coefficients.)

Solution: To solve this problem, we will use the fact that the number of solutions to the equation

$$z_1 + z_2 + \ldots + z_r = q$$

where $z_1, z_2, \ldots, z_r$ and $q$ are all non-negative integers, is $C(q + r - 1, r - 1)$.

Since $x_4$ must be positive multiple of 8, it must assume values from the set $\{8, 16, 24, 32, \ldots\}$. Since all the variables are non-negative integers and the total should be 31, there is no solution when $x_4$ takes on any value $\geq 32$. In other words, $x_4$ must assume a value from the set $\{8, 16, 24\}$. We compute the number of solutions in each of these three cases. The required answer is the sum of the numbers obtained in the three cases.

Case 1: $x_4 = 8$. In this case, the given equation becomes

$$x_1 + x_2 + x_3 = 23.$$  

Using the above formula, the number of solutions is $C(23 + 3 - 1, 3 - 1) = C(25, 2)$. 

Case 2: $x_4 = 16$. In this case, the given equation becomes

$$x_1 + x_2 + x_3 = 15.$$
Using the above formula, the number of solutions is \( C(15 + 3 - 1, 3 - 1) = C(17, 2) \).

Case 3: \( x_4 = 24 \). In this case, the given equation becomes

\[
x_1 + x_2 + x_3 = 7.
\]

Again, using the above formula, the number of solutions is \( C(7 + 3 - 1, 3 - 1) = C(9, 2) \).

Thus, the answer to the problem is \( C(25, 2) + C(17, 2) + C(9, 2) \). (This expression can be simplified to 472.)

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**Problem 10:** What is the coefficient of \( x^{90}y^{7}z^{110} \) in the expansion of \( (7x - 3y - z)^{207} \)? You need not simplify your answer.

**Solution:** Let \( q = 7x - 3y \). Thus, the function to be expanded is \( (q - z)^{207} \). In this expansion, by the Binomial Theorem, the term \( T_1 \) containing \( z^{110} \) is given by

\[
T_1 = C(207, 97) q^{97} (-z)^{110} = C(207, 97) q^{97} z^{110}
\]

(5)

Now, consider the term \( q^{97} = (7x - 3y)^{97} \) occurring in \( T_1 \). When \( (7x - 3y)^{97} \) is expanded using the Binomial Theorem, the term \( T_2 \) which contains \( x^{90} y^{7} \) is given by

\[
T_2 = C(97, 90) (7x)^{90} (-3y)^7 = -C(97, 90) 7^{90} 3^7 x^{90} y^{7}.
\]

(6)

From the equations for \( T_1 \) and \( T_2 \), it follows that the term containing \( x^{90} y^{7} z^{110} \) in the expansion of \( (7x - 3y - z)^{207} \) is given by

\[
-C(207, 97) C(97, 90) 7^{90} 3^7 x^{90} y^{7} z^{110}
\]

Therefore, the required coefficient is

\[-C(207, 97) C(97, 90) 7^{90} 3^7.\]

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**Problem 11:** Consider the infinite sequence of integers \( f_0, f_1, f_2, \ldots, \) defined by \( f_0 = f_1 = 6 \) and \( f_n = 2f_{n-1} + 3f_{n-2} \) for all \( n \geq 2 \). Use induction on \( n \) to prove that for all \( n \geq 0 \), \( f_n = 3 \left[ 3^n + (-1)^n \right] \).

**Solution:** Since \( f_n \) is defined using \( f(n - 1) \) and \( f(n - 2) \), we will use the strong form of induction to prove this result.

**Basis:** We verify the basis for \( n = 0 \) and \( n = 1 \).

Case 1: \( n = 0 \). Here \( f_0 = 6 \) (given). For \( n = 0 \), the value of the expression \( 3 \left[ 3^n + (-1)^n \right] \) is \( 3 \left[ 3^0 + (-1)^0 \right] = 3(1 + 1) = 6 \). Thus, the basis is true for \( n = 0 \).

Case 2: \( n = 1 \). Here \( f_1 = 6 \) (given). For \( n = 1 \), the value of the expression \( 3 \left[ 3^n + (-1)^n \right] \) is \( 3 \left[ 3^1 + (-1)^1 \right] = 3(3 - 1) = 6 \). Thus, the basis is true for \( n = 1 \) as well.

**Induction Hypothesis:** Assume that for some \( k \geq 1 \) and all \( r, 0 \leq r \leq k \), \( f_r = 3 \left[ 3^r + (-1)^r \right] \).

**To prove:** \( f_{k+1} = 3 \left[ 3^{k+1} + (-1)^{k+1} \right] \).

**Proof:** Since \( k \geq 1 \), \( k + 1 \geq 2 \). So, we can apply the recursive definition to \( f_{k+1} \) to get

\[
f_{k+1} = 2f_k + 3f_{k-1}
\]

(7)

To make it easier to follow the calculations, we will consider each term on the right side of Equation (7) separately. The first term is \( 2f_k \). Now,
\[ f_k = 2 \left[ 3^k + (-1)^k \right] \quad \text{(Inductive hypothesis for } f_k) \]
\[ = 2 \cdot 3^{k+1} + 6 (-1)^k \]
\[ = 2 \cdot 3^{k+1} - 6 (-1)^{k+1} \]

The second term is \( 3 f_{k-1} \). Now,
\[ 3 f_{k-1} = 3 \left[ 3^{k-1} + (-1)^{k-1} \right] \quad \text{(Inductive hypothesis for } f_{k-1}) \]
\[ = 3^{k+1} + 9 (-1)^{k-1} \]
\[ = 3^{k+1} + 9 (-1)^{k+1} \]

Substituting the expressions for \( 2 f_k \) and \( 3 f_{k-1} \) in Equation (7), we get
\[ f_{k+1} = 2 \cdot 3^{k+1} - 6 (-1)^{k+1} + 3^{k+1} + 9 (-1)^{k+1} \]
\[ = 3 \cdot 3^{k+1} + 3 (-1)^{k+1} \]
\[ = 3 \left[ 3^{k+1} + (-1)^{k+1} \right] \]

The last equation shows that \( f_{k+1} \) has the required form, and this completes the proof.

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**Problem 12:** Consider a square whose side has length 1. Suppose \( S \) is an arbitrarily chosen set of 5 points from this square. Prove that \( S \) must contain two points whose distance is at most \( 1/\sqrt{2} \).

**Proof:** Let \( Q \) denote the given square each of whose sides has length 1. Divide \( Q \) into 4 subsquares \( Q_1, Q_2, Q_3 \) and \( Q_4 \), each of whose sides has length \( 1/2 \) as shown below.

![Diagram](image)

In each subsquare, the maximum distance between any pair of points is the length of a diagonal. By Pythagoras Theorem, the length of any diagonal in a subsquare is \( \sqrt{(1/2)^2 + (1/2)^2} = 1/\sqrt{2} \). Thus, we have the following observation.

**Observation:** For any subsquare \( Q_i \) \( (i \in \{1, 2, 3, 4\}) \), the maximum distance between a pair of points in \( Q_i \) is \( 1/\sqrt{2} \).

Now, consider any set \( S = \{p_1, p_2, p_3, p_4, p_5\} \) of 5 points chosen from \( Q \). Consider a function that maps each \( p_i \) to one of the subsquare in which \( p_i \) lies. (If \( p_i \) lies on a line shared by two subsquares or \( p_i \) is the center of \( Q \) that is shared by all the subsquares, the function maps \( p_i \) to one of the subsquares arbitrarily.) Since there are 5 points and only 4 subsquares, by the pigeon hole principle, the function must map at least two points of \( S \), say \( p_x \) and \( p_y \), to the same subsquare \( Q_i \). By the observation mentioned above, the distance between \( p_x \) and \( p_y \) is at most \( 1/\sqrt{2} \), and this completes the proof.