You may find the following definitions and remarks useful. The problems are on page 3. You should try all of the problems. All problems will be weighted equally.

1. The symbol “∀” means “for all”; the symbol “∃” means “there exists”; the symbol “∃!” means “there exists a unique”;

2. The symbols ¬, ∧, ∨ and → denote respectively the negation (NOT), and, or and implication operators of propositional calculus.

3. The symbol “∅” denotes the empty set. For a set A, the power set of A (denoted by \(\mathcal{P}(A)\)) is the set of all subsets of A.

4. For an alphabet \(\Sigma\) (i.e., a finite set of symbols), \(\Sigma^*\) denotes the set of all finite length strings that can be formed using the symbols in \(\Sigma\). The symbol \(\lambda\) denotes the null string (i.e., the string of length zero).

5. (a) Let \(f\) be a function from a set \(A\) to a set \(B\). (Notation: \(f : A \rightarrow B\).) The function \(f\) is one-to-one if \(\forall x, y \in A, f(x) = f(y) \Rightarrow x = y\); \(f\) is onto if \(\forall y \in B, \exists x \in A\) such that \(y = f(x)\). If \(f\) is both one-to-one and onto, then \(f^{-1}\) is the function from \(B\) to \(A\) defined by \(f^{-1}(y) = x\), where \(f(x) = y\).

(b) A function on a set \(A\) is a function from \(A\) to \(A\).

(c) If \(f_1\) is a function from \(A\) to \(B\) and \(f_2\) is a function from \(B\) to \(C\), the composition \(f_2 \circ f_1\) is a function from \(A\) to \(C\) defined by \(f_2 \circ f_1(x) = f_2(f_1(x))\).

6. Let \(U\) be a non-empty set and let \(R\) be a binary relation on \(U\) (i.e., \(R \subseteq U \times U\)).

(a) \(R\) is reflexive if \(\forall x \in U, (x, x) \in R\).

(b) \(R\) is symmetric if \(\forall x, y \in U, (x, y) \in R \Rightarrow (y, x) \in R\).

(c) \(R\) is antisymmetric if \(\forall x, y \in U, [(x, y) \in R \text{ and } (y, x) \in R] \Rightarrow x = y\).

(d) \(R\) is transitive if \(\forall x, y, z \in U, [(x, y) \in R \text{ and } (y, z) \in R] \Rightarrow (x, z) \in R\).

(e) The reflexive closure (respectively, symmetric closure and transitive closure) of \(R\) is the smallest binary relation \(S\) on \(U\) containing \(R\) such that \(S\) is reflexive (respectively, symmetric and transitive).

(f) An equivalence relation on \(U\) is a binary relation which is reflexive, symmetric and transitive.
(g) A **partial order** on $U$ is a binary relation which is reflexive, antisymmetric and transitive.

7. (a) Let $R$ be a relation from $A$ to $B$ (i.e., $R \subseteq A \times B$). The **converse** of $R$, denoted by $R^{-1}$, is defined by $R^{-1} = \{(y,x) \mid (x,y) \in R\}$.

(b) If $R$ is a relation from $A$ to $B$ and $S$ is a relation from $B$ to $C$, the **composition** of $R$ with $S$, denoted by $R \circ S$, is a relation from $A$ to $C$ defined by

$$R \circ S = \{(x,y) \mid \text{for some } z \in B, (x,z) \in R \text{ and } (z,y) \in S\}.$$ 

(c) When $R$ is a relation on a set $A$, $R^2$ denotes $R \circ R$, $R^3$ denotes $R \circ R \circ R$, etc. In general, for all integers $n \geq 0$, $R^n$ is defined as follows.

$$R^0 = \{(x,x) \mid x \in A\}, \quad \text{and} \quad R^n = R^{n-1} \circ R \quad \text{for } n > 0.$$ 

8. The **Binomial Theorem** states that for any integer $n \geq 0$,

$$(a + b)^n = \sum_{i=0}^{n} C(n,i) a^{n-i} b^i$$

where $C(n,i) = \frac{n!}{(n-i)!i!}$ $(0 \leq i \leq n)$, are the binomial coefficients.

9. Let $x$ be a real number. The **floor** of $x$, denoted by $\lfloor x \rfloor$, is the largest integer $\leq x$. The **ceiling** of $x$, denoted by $\lceil x \rceil$, is the smallest integer $\geq x$.

10. An undirected graph $G$ with vertex set $V$ and edge set $E$ is denoted by $G(V,E)$. The sets $V$ and $E$ are both finite. For any vertex $v \in V$, the **degree** of $v$ is the number of edges incident on $v$. A **path** $p$ from a vertex $u$ to a vertex $v$ in $G$ is a sequence of distinct vertices $< u, v_1, v_2, ..., v_{r-1}, v_r, v >$ such that the edges $\{u,v_1\}, \{v_1,v_2\}, ..., \{v_{r-1},v_r\}, \{v_r,v\}$ are all in $E$. A **cycle** in $G$ is a path that begins and ends at the same vertex. Graph $G(V,E)$ is **connected** if there is a path between every pair of vertices in $V$; $G$ is **acyclic** if it does not contain any cycles.
1. Specify two different predicates \( P(x) \) and \( Q(x) \) over the set of positive integers so that the proposition \( \forall x \ (P(x) \land Q(x)) \) is false while \( \exists x \ (P(x) \land Q(x)) \) is true. Your answer should contain a clear definition of the predicates \( P(x) \) and \( Q(x) \) and an explanation of why, for your choice of predicates, \( \forall x \ (P(x) \land Q(x)) \) is false and \( \exists x \ (P(x) \land Q(x)) \) is true.

2. Suppose \( p, q \) and \( r \) are propositions such that \( p \to q \), \( \lnot p \to r \) and \( r \to (p \lor q) \) are all true. Show that \( q \) is true.

3. Recall that a binary propositional operator \( * \) is associative if for all propositions \( a, b \) and \( c \), the propositional forms \( a * (b * c) \) and \( (a * b) * c \) have the same truth value. Also recall that the NAND operator \( (\lnot \land) \) is defined by \( a \land b = \lnot (a \land b) \). Is the NAND operator associative? Justify your answer.

4. Suppose \( A \) and \( B \) are sets such that \( |A| = 6 \), \( |B| = 2 \) and \( |A \cap B| = 1 \). Calculate \( |P(A \cup B)| \). (Recall that for a set \( X \), \( P(X) \) denotes the powerset of \( X \).)

5. Let \( A = \{1, 2, 3, 4\} \) and \( B = \{a, b, c, d, e\} \). How many functions from \( A \) to \( B \) are either one-to-one or map the element 1 to \( c \)? (You need not simplify your answer.)

6. Let \( X = \{a, b, c, d, e\} \). Let us call a binary relation \( R \) on \( X \) special if it satisfies all of the following conditions: (i) \( R \) is reflexive, (ii) \( R \) is symmetric and (iii) \( R \) contains the pair \((a, b)\). Find the number of special binary relations on \( X \). You need not simplify your answer.

7. Let \( Y = \{1, 2, 3, 4\} \). Consider the binary relation \( R \) on \( Y \) defined by

\[
R = \{(1, 2), (2, 1), (2, 3), (2, 4), (4, 3), (4, 4)\}.
\]

Find the transitive closure of \( R \).

8. Let \( N^+ \) denote the set of positive integers (i.e., integers that are strictly greater than 0). Consider the function \( f \) from \( N^+ \) to \( N^+ \) defined as follows:

\[
f(x) = 1 + \text{the number of 9's in the decimal representation of } x.
\]

For example, \( f(1) = 1 \), \( f(293) = 2 \), \( f(1929) = 3 \).

(a) Is \( f \) one-to-one? Justify your answer.

(b) Is \( f \) onto? Justify your answer.

9. Let \( R \) denote the set of real numbers. Consider the function \( f \) from \( R \times R \) to \( R \times R \) defined by \( f(x, y) = (x + y, x - y) \). Find \( f^{-1} \). (You may assume without proof that \( f \) is a bijection.)

10. Find the number of solutions to the equation \( x_1 + x_2 + x_3 + x_4 + x_5 = 37 \), where \( x_1, x_2, x_3, x_4 \) and \( x_5 \) are non-negative integers, \( x_2 \geq 8 \), \( x_3 \geq 7 \), \( x_4 \geq 2 \) and \( x_5 < 4 \). (You may leave the answer as an expression consisting of binomial coefficients.)

11. Recall that a bit string is a string composed of characters 0 and 1. Let us call a bit string \( s \) interesting if it satisfies all of the following conditions: (i) \( s \) has length 23, (ii) \( s \) starts with 1110, (iii) \( s \) ends with 00110 and (iv) \( s \) has 0 as its middle bit. Find the number of bit strings that are interesting. (You need not simplify your answer.)

12. Use induction on \( n \) to prove the following for all \( n \geq 2 \): If \( A_1, A_2, \ldots, A_n \) are subsets of a universal set \( U \), then \( A_1 \cap A_2 \cap \ldots \cap A_n = \overline{A_1} \cup \overline{A_2} \cup \ldots \cup \overline{A_n} \). You can assume without proof that for any two subsets \( X \) and \( Y \) of the universal set \( U \), \( \overline{X \cap Y} = \overline{X} \cup \overline{Y} \).