Quantum Computing

Samuel J. Lomonaco, Jr.
Dept. of Comp. Sci. & Electrical Engineering
University of Maryland Baltimore County
Baltimore, MD 21250
Email: Lomonaco@UMBC.EDU
WebPage: http://www.csee.umbc.edu/~lomonaco

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- L-O-O-P Fund.

Overview

Four Talks

- A Rosetta Stone for Quantum Computation
- Quantum Hidden Subgroup Algorithms
- An Entangled Tale of Quantum Entanglement

These talks are available at:

http://www.csee.umbc.edu/~lomonaco
A Rosetta Stone for Quantum Computation

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& many more

Why Quantum Computation

- Limits of small scale integration technology to be reached 2010-2020
- No Longer! Moore's Law, i.e., every year, double the computing power at half the price. No Longer!
- A whole new industry will be built around the new & emerging quantum technology

Collision Course

Quantum Computation

Math
Physics
Comp Sci
EE
Multi-Disciplinary
The Classical World

Classical Shannon Bit

Classical Bits Can Be Copied

The Quantum World

Quantum Bit Qubit

Quantum Bit Qubit

In

Out

Copying Machine

Indecisive Individual

Can be both 0 & 1 at the same time !!!
**Quantum Representations of Qubits**

**Example 1.** A spin-\(\frac{1}{2}\) particle

Spin Up
\[1\]

Spin Down
\[0\]

**Quantum Representations of Qubits (Cont.)**

**Example 2.** Polarization States of a Photon

\[1 = \begin{pmatrix} \sqrt{2} \end{pmatrix}, \quad 0 = \begin{pmatrix} 1 \end{pmatrix}\]

or

\[1 = \begin{pmatrix} 1 \end{pmatrix}, \quad 0 = \begin{pmatrix} 0 \end{pmatrix}\]

**Where does a Qubit live?**

A Hilbert Space is a vector space over \(\mathbb{C}\) together with an inner product \((\cdot, \cdot): H \times H \to \mathbb{C}\) such that

1. \(\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle \quad \text{and} \quad \langle \alpha u, v \rangle = \alpha \langle u, v \rangle \)
2. \(\langle u, v \rangle = \overline{\langle v, u \rangle}\)
3. \(\langle u, u \rangle \geq 0 \quad \text{and} \quad \langle u, u \rangle = 0 \iff u = 0\)
4. \(\forall\) Cauchy seq \(u_n, u_{n+1}, \ldots\) in \(H\), \(\lim_{n \to \infty} u_n \in H\)

The elements of \(H\) will be called kets, and will be denoted by \(\label{label1}\).

**A Qubit** is a quantum system whose state is represented by a Ket lying in a 2-D Hilbert Space \(H\).

**Superposition of States**

A typical Qubit is

\[\alpha_0|0\rangle + \alpha_1|1\rangle\]

where \(\alpha_0^2 + |\alpha_1|^2 = 1\)

The above Qubit is in a Superposition of states \(|0\rangle\) and \(|1\rangle\).

It is simultaneously both \(|0\rangle\) and \(|1\rangle\) !!!

**“Collapse” of the Wave Function**

\[\alpha_0|0\rangle + \alpha_1|1\rangle =\]

Observer

Qubit

Whoosh !!!

Prob \(\propto |\alpha|^2\)

\(|i\rangle\)
Kets as Column Vectors over \( \mathbb{C} \)

Let \( H \) be a 2-D Hilbert space with orthonormal basis \( |0\rangle, |1\rangle \).

In this basis, each ket can be thought of as a column vector. For example,

\[
|0\rangle = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{and} \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\]

And in general, we have

\[
|\psi\rangle = a|0\rangle + b|1\rangle = a \begin{pmatrix} 0 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}
\]

Tensor Product of Hilbert Spaces

The tensor product of two Hilbert spaces \( H \) and \( K \) is the "simplest" Hilbert space such that the map \( (h, k) \mapsto h \otimes k \)

\[H \times K \rightarrow H \otimes K\]

is bilinear, i.e., such that

\[
(h_1 + h_2) \otimes k = h_1 \otimes k + h_2 \otimes k \\
(h \otimes (k_1 + k_2)) = h \otimes k_1 + h \otimes k_2 \\
(\lambda h) \otimes k = \lambda (h \otimes k)
\]

We define the action of \( \mathbb{C} \) on \( H \otimes K \) as

\[
\lambda (h \otimes k) = (\lambda h) \otimes k = h \otimes (\lambda k)
\]

In other words, \( H \otimes K \) is constructed in the simplest non-trivial way such that:

\[
(h_1 + h_2) \otimes k = h_1 \otimes k + h_2 \otimes k \\
h \otimes (k_1 + k_2) = h \otimes k_1 + h \otimes k_2 \\
(\lambda h) \otimes k = \lambda (h \otimes k), \forall \lambda \in \mathbb{C}
\]

Kronecker (Tensor) Product of Matrices

\[
A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}
\]

The Kronecker (tensor) product is defined as:

\[
A \otimes B = \begin{pmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\ a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} \end{pmatrix}
\]

Representing Integers in Quantum Computation

Let \( H_2 \) be a 2-D Hilbert space with orthonormal basis \( |0\rangle, |1\rangle \).

Then \( H = \bigotimes_{n=0}^{n-1} H_2 \) is a \( 2^n \)-D Hilbert space with induced orthonormal basis

\[
|0\rangle, |01\rangle, |001\rangle, |0001\rangle, \ldots, |111\rangle
\]

where we are using the convention

\[
b_{n-1}b_{n-2} \cdots b_1b_0 = b_{n-1} \otimes b_{n-2} \otimes \cdots \otimes b_1 \otimes b_0.
\]
Representing Integers in Quantum Computation

So in the $2^n$-D Hilbert space with induced orthonormal basis

$$|0\rangle, \ldots, |01\rangle, |010\rangle, \ldots, |1\rangle$$

we represent the integer $m$ with binary expansion

$$m = \sum_{j=0}^{n-1} m_j 2^j, \quad m_j = 0 \text{ or } 1, \forall j$$

as the ket $|m\rangle = |m_{n-1}m_{n-2}\cdots m_1m_0\rangle$

For example,

$$23 = 010111$$

Indexing Convention for Matrices

The indices of matrices start at 0, not 1. For example, in $H_2 \otimes H_2 \otimes H_2$

$$|5\rangle = |101\rangle = \left(\frac{1}{\sqrt{2}}\right)^3 \left|0\right\rangle \otimes \left|1\right\rangle \otimes \left|0\right\rangle = \left(\frac{1}{\sqrt{2}}\right)^3 \left(0 \otimes 1 \otimes 0\right) = \begin{cases} 0 \quad \text{index } = 0 \\ 0 \quad \text{index } = 1 \\ 0 \quad \text{index } = 2 \\ 0 \quad \text{index } = 3 \\ 0 \quad \text{index } = 4 \\ 0 \quad \text{index } = 5 \\ 0 \quad \text{index } = 6 \\ 0 \quad \text{index } = 7 \end{cases}$$

The Qubit Village

- The Qubits in Qubit Village collectively live in $H_1 \otimes H_2 \otimes \cdots \otimes H_n = \bigotimes_{j=1}^n H_j$
- The populace of Qubit Village is

$$|\text{Population}\rangle = |\Psi_1\rangle \otimes |\Psi_2\rangle \otimes \cdots \otimes |\Psi_n\rangle \in \bigotimes_{j=1}^n H_j$$

- Other names for the populace of Qubit Village

$$|\text{Population}\rangle = |\Psi_1\rangle |\Psi_2\rangle \cdots |\Psi_n\rangle = |\Psi_1\Psi_2\cdots\Psi_n\rangle$$

Massive Parallelism

Example. For $j = 1, 2, \ldots, n$, let $|\Psi_j\rangle = |0 + 1\rangle = |0\rangle + |1\rangle$ Then

$$|\Psi_j\rangle |\Psi_j\rangle |\Psi_j\rangle = \left(\frac{1}{\sqrt{2}}\right)^3 \left(|0\rangle + |1\rangle\right) \otimes \left(|0\rangle + |1\rangle\right) \otimes \left(|0\rangle + |1\rangle\right)$$

$$= \frac{1}{\sqrt{2}} \left(|0\rangle + |1\rangle\right) \otimes \left(|0\rangle + |1\rangle\right) \otimes \left(|0\rangle + |1\rangle\right)$$

$$= \frac{1}{\sqrt{2}} \left(|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle + |111\rangle\right)$$

Therefore, the $n$-qubit register contains all $n$-bit binary numbers simultaneously!

Activities in Quantum Village

All activities in Quantum Village are Unitary transformations

$$|\Psi_0\rangle \xrightarrow{U \text{ at time } t=0} |\Psi_1\rangle \xrightarrow{U \text{ at time } t=1} |\Psi_2\rangle$$

where a unitary transformation is one such that

$$U^T \cdot U = I = UU^T$$
Another Activity in Quantum Village: Measurement

Connecting Quantum Village to the Classical World

Another Activity in Quantum Village: Measurement

What does our observer actually observe?

Observables = Hermitian Operators

Observables (Cont.)

Observables (Cont.)

Whoosh!
Example: Pauli Spin Matrices

Consider the following observables, called the Pauli Spin matrices:
\[ \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]

which can readily be checked to be Hermitian.

E.g.,
\[ \sigma_x = \begin{pmatrix} 0 & -1 \\ i & 0 \end{pmatrix} \]

The respective eigenvalues and eigenkets of these matrices are listed in the table below:

<table>
<thead>
<tr>
<th>Eigenvalue</th>
<th>( \sigma_x )</th>
<th>( \sigma_y )</th>
<th>( \sigma_z )</th>
</tr>
</thead>
<tbody>
<tr>
<td>+1</td>
<td>( (0+1)/\sqrt{2} )</td>
<td>( (0+i1)/\sqrt{2} )</td>
<td>( 0 )</td>
</tr>
<tr>
<td>-1</td>
<td>( (0-1)/\sqrt{2} )</td>
<td>( (0-i1)/\sqrt{2} )</td>
<td>( 1 )</td>
</tr>
</tbody>
</table>

Important Feature of Quantum Mechanics

It is important to mention that:

**We cannot completely control the outcome of quantum measurement**

The No Cloning Theorem

Definition. Let \( H \) be a Hilbert space. Then a quantum replicator consists of an auxiliary Hilbert space \( H' \), a fixed state \( \psi_0 \) \( \in \) \( H' \) (called the initial state of the replicator), and a unitary transformation

\[ U : H \otimes H' \otimes H \rightarrow H \otimes H' \otimes H \]

such that, for some fixed state \( \chi_0 \) \( \in \) \( H \),

\[ U(\psi_0 \otimes \chi_0 \otimes \mathrm{blank}) = (\psi_0 \otimes \psi_0 \otimes \psi_0) \]

for all states \( \psi_0 \) \( \otimes \) \( \chi_0 \) \( \in \) \( H \otimes H' \) (called the replicator state after replication of \( \psi_0 \) depends on \( \chi_0 \) .
Observing Entangled Qubits

• Cloning is inherently non-linear
• Quantum mechanics is inherently linear
• Ergo, quantum replicators do not exist

Something is Missing from Quantum Mechanics.

Separate Identity
No Longer Entangled

• Not Entangled
• Separate

Entangled Qubits

\[ |0\rangle \otimes |0\rangle \]

\[ U \]

\[ \frac{1}{\sqrt{2}} (|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle) \]

Not Entangled
Separate

EPR Pair

\[ \frac{1}{\sqrt{2}} (|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle) \]

Einstein
Podolsky
Rosen

Bah! Humbug!

Something is Missing from Quantum Mechanics.
There Must Exist Hidden Variables

Hidden Variable Theory vs Quantum Mechanics

Prob = 1/2
The forces of nature are local interactions.

- Mediated by another entity, e.g., gravitons, photons, etc.
- Propagate no faster than the speed of light $c$
- Strength drops off with distance

Einstein, Podolsky, Rosen

Object So

Vehemently?

Why did

Score So Far

- HVT Score = 0
- QM Score = 1

Forces of Nature Are Local Interactions

All the forces of nature (i.e., gravitational, electromagnetic, weak, & strong forces) are local interactions.

- Mediated by another entity, e.g., gravitons, photons, etc.
- Propagate no faster than the speed of light $c$
- Strength drops off with distance

The EPR Perspective

- The forces of nature are local interactions
- Spacelike regions of space are physically independent

All perfectly reasonable assumptions!
Instantly, both qubits are determined!

\[
\frac{\left( 0\ 1 \right) - \left( 1\ 0 \right)}{\sqrt{2}}
\]

Spacelike distance

Meas. Blue Qubit

Red Qubit

\[ 0 \otimes 1 \]

\[ 1 \otimes 0 \]

No Local Interaction!

- No force of any kind
  - Not mediated by anything
- Acts instantaneously
  - Faster than light
- Strength does not drop off with distance
  - Full strength at any distance

Yet, still consistent with General Relativity!

Quantum Entanglement appears to pinpoint the weirdness of Quantum Mechanics

Properties of Qubits Useful for Quantum Computation

- Properties of Quantum Computer Data
  - Qubits can exist in a superposition of states
  - Qubits can be entangled
- Quantum Computer Instructions
  - Qubits “collapse” upon measurement
  - Qubits are transformed by unitary transformations

Quantum Teleportation

An Application of Quantum Entanglement

Teleportation: Transferring an object between two locations by a process of:

- Dissociation to obtain info
  - Scanned to extract sufficient info. to recreate original
- Information Transmission
- Reconstruction from info
  - Exact replica is re-assembled at destination out of locally available material

Net Effects:

- Destruction of original object
- Creation of an exact replica at the intended destination.

Oxford Unabridged Dictionary

Teleportation: Transferring an object between two locations by a process of:

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Net Effects:

- Destruction of original object
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**Quantum Teleporting Manual**

**Step 1 (Loc. A): Preparation**
- At location A, construct an EPR pair of qubits (qubits #2 & #3) in $\mathcal{H}_2 \otimes \mathcal{H}_3$.

\[
\begin{align*}
|00\rangle & \xrightarrow{\text{Unitary Matrix}} \frac{|01\rangle - |10\rangle}{\sqrt{2}} \\
\mathcal{H}_2 \otimes \mathcal{H}_3 & \xrightarrow{} \mathcal{H}_2 \otimes \mathcal{H}_3
\end{align*}
\]
- Physically transport entangled qubit #3 from Loc. A to Loc. B

**Result**
- Loc. A & Loc. B share an EPR pair, i.e.,
  - Qubit #2 is at Loc. A
  - Qubit #3 is at Loc. B
  - Qubits #2 & #3 are entangled
- The state of all three qubits is:
  \[
  |\Phi\rangle = (a |0\rangle + b |1\rangle) \left( \frac{|01\rangle - |10\rangle}{\sqrt{2}} \right) \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3
  \]

**A Little Algebraic Manipulation**
- **Current state** of all three qubits:
  \[
  |\Phi\rangle = (a |0\rangle + b |1\rangle) \left( \frac{|01\rangle - |10\rangle}{\sqrt{2}} \right) \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3
  \]
- **The Bell basis** of $\mathcal{H}_1 \otimes \mathcal{H}_2$ is:
  \[
  \begin{align*}
  |\psi^A\rangle &= (|10\rangle - |01\rangle) / \sqrt{2} \\
  |\psi^B\rangle &= (|10\rangle + |01\rangle) / \sqrt{2} \\
  |\psi^C\rangle &= (|00\rangle - |11\rangle) / \sqrt{2} \\
  |\psi^D\rangle &= (|00\rangle + |11\rangle) / \sqrt{2}
  \end{align*}
  \]
- Re-express $|\Phi\rangle$ in terms of the Bell basis as:
  \[
  |\Phi\rangle = \frac{1}{2} \left| \psi^A \right> (-a |0\rangle - b |1\rangle) \\
  + |\psi^B \rangle (-a |0\rangle + b |1\rangle) \\
  + |\psi^C \rangle (a |1\rangle + b |0\rangle) \\
  + |\psi^D \rangle (a |1\rangle - b |0\rangle)
  \]
- Let $U : \mathcal{H}_1 \otimes \mathcal{H}_2 \rightarrow \mathcal{H}_1 \otimes \mathcal{H}_2$ be the unitary transformation:
  \[
  \begin{align*}
  |\psi^A\rangle &\rightarrow |00\rangle \\
  |\psi^B\rangle &\rightarrow |01\rangle \\
  |\psi^C\rangle &\rightarrow |10\rangle \\
  |\psi^D\rangle &\rightarrow |11\rangle
  \end{align*}
  \]
Step 2. (Loc. A): Apply $U \otimes I : \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3 \rightarrow \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$ to the three qubits. Thus, under $U \otimes I$ the state $|\Phi\rangle$ of all three qubits becomes:

$$|\Phi\rangle = \frac{1}{2} \left( |000\rangle (-a|0\rangle - b|1\rangle)
+ |011\rangle (-a|0\rangle + b|1\rangle)
+ |100\rangle (a|1\rangle + b|0\rangle)
+ |111\rangle (a|1\rangle - b|0\rangle) \right)$$


Step 4. (Loc. A): Send via a classical communication channel the result of the measurement to Loc. B.

**Result**

Unknown qubit #1 has been disassembled and the info read (two classical bits) is sent to Loc. B.

### Table: Unitary Transformation

<table>
<thead>
<tr>
<th>Rec. Bits</th>
<th>$U^{(i,j)}$</th>
<th>Effect on Qubit #3</th>
</tr>
</thead>
<tbody>
<tr>
<td>00</td>
<td>$U^{(00)}$</td>
<td>$a</td>
</tr>
<tr>
<td>01</td>
<td>$U^{(01)}$</td>
<td>$a</td>
</tr>
<tr>
<td>10</td>
<td>$U^{(10)}$</td>
<td>$a</td>
</tr>
<tr>
<td>11</td>
<td>$U^{(11)}$</td>
<td>$a</td>
</tr>
</tbody>
</table>

Step 5. (Loc. B): Use the two classical bits $(i,j)$ of received information to select a unitary transformation $U^{(i,j)}$ of $\mathcal{H}_3$ from the table.

**Result**

Qubit #3 at Loc. B now has the original state that qubit #1 had at Loc. A before it was disassembled, i.e.,

$$a|0\rangle + b|1\rangle$$

Step 6. (Loc. B): Apply the selected Unitary transformation $U^{(i,j)}$ to qubit #3.

### More Dirac Notation

Let $H' = \text{Hom}(H, \mathbb{C})$.

We call the elements of $H'$ Bra's, and denote them as

\[ \langle \text{label} \rangle \]
More Dirac Notation

There is a dual correspondence between $H^*$ and $H$

$$|\psi\rangle \leftrightarrow \langle \psi|$$

There exists a bilinear map $H^* \times H \to \mathbb{C}$ defined by

$$(\langle \psi_1|)(|\psi_2\rangle) \in \mathbb{C}$$

which we more simply denote by

$$\langle \psi_1| \psi_2 \rangle$$

Bra's & Ket's as Adjoints of One Another

The dual correspondence

$$H \leftrightarrow H^*$$

is given by

$$\begin{pmatrix} a \\ b \end{pmatrix} = a \langle 0| + b \langle 1| \leftrightarrow \begin{pmatrix} a \\ b \end{pmatrix} = a |0\rangle + b |1\rangle$$

and is called the adjoint

$|\psi\rangle \langle \psi| \text{ as a Matrix Outerproduct}$

If

$$|\psi_1\rangle = a |0\rangle + b |1\rangle$$
$$|\psi_2\rangle = c |0\rangle + d |1\rangle$$

then $\langle \psi_1| \psi_2 \rangle$ is the linear transformation

$$H \rightarrow H$$
$$\psi \rightarrow \langle \psi_1| \psi_2 \rangle$$

which, when written in matrix notation, becomes the matrix outerproduct

$$\begin{pmatrix} a \\ b \end{pmatrix} \cdot \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} ac & ad \\ bc & bd \end{pmatrix}$$

Bra's as Row Vectors over $\mathbb{C}$

Let $H$ be a 2-D Hilbert space with orthonormal basis $|0\rangle, |1\rangle$

and let $H^* = Hom(H, \mathbb{C})$

be the corresponding dual Hilbert space with corresponding dual basis $\langle 0|, \langle 1|$

Then with respect to this basis, we have

$$\langle 0| a + \langle 1| b = (a, b)$$

If

$$|\psi_1\rangle = a |0\rangle + b |1\rangle$$
$$|\psi_2\rangle = c |0\rangle + d |1\rangle$$

then the bracket product becomes

$$\langle \psi_1| \psi_2 \rangle = (\langle 0| a + \langle 1| b)(c |0\rangle + d |1\rangle)$$

$$= (a, b) \cdot \begin{pmatrix} c \\ d \end{pmatrix} = ac + bd$$

Let $H$ be an N-D Hilbert space with orthonormal basis $|0\rangle, |1\rangle, \ldots, |N-1\rangle$

If we use the convention that matrix indices begin at 0, then the matrix of the linear transformation is an $N \times N$ matrix consisting of all zeroes with the exception of entry $(m, k)$ which is 1

For example if $N=4$, then

$$2/3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
Density Operators & Mixed Ensembles

Two Ways to Represent Quantum States

Example. We have seen pure ensembles, i.e., pure states, such as

\[ \langle \psi \rangle, \quad \text{Prob} \quad 1 \]

Problem. Certain types of quantum states are difficult to represent in terms of kets

\[ \text{Prob} \quad \langle \psi \rangle, \psi_2, \ldots, \psi_k \]

Two Ways to Represent Quantum States

Johnny von Neumann suggested that we use the following operator to represent a state:

\[ \rho = p_1 \langle \psi_1 \rangle \psi_1 + p_2 \langle \psi_2 \rangle \psi_2 + \ldots + p_k \langle \psi_k \rangle \psi_k \]

\( \rho \) is called a density operator. It is a Hermitian positive definite operator of trace 1.

For the pure ensemble

\[ \text{Ket} \quad \langle \psi \rangle, \quad \text{Prob} \quad 1 \]

\[ \rho = \langle \psi \rangle \psi \]

Mixed Ensemble

Two Ways to Represent Quantum States

Example. Consider the following state for which we have incomplete knowledge, called a mixed ensemble:

\[ \text{Ket} \quad \langle \psi_1 \rangle, \psi_2, \ldots, \psi_k \quad \text{Prob} \quad p_1, p_2, \ldots, p_k \]

where

\[ p_1 + p_2 + \ldots + p_k = 1 \]

Example. If for example,

\[ \psi = a \langle 0 \rangle + b \langle 1 \rangle \]

then

\[ \rho = (a^* b) \begin{pmatrix} a^2 & ab \\ b^* a & b^2 \end{pmatrix} = \begin{pmatrix} |a|^2 & ab \\ b^* a & |b|^2 \end{pmatrix} \]
Two Ways to Represent Quantum States

On the other hand, the mixed ensemble

$$\rho = \frac{3}{4} \left( \begin{array}{cc} 0 - i \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{array} \right) = \frac{3}{8} \left( \begin{array}{cc} i/8 & 5/8 \\ 5/8 & 3/8 \end{array} \right)$$

is the mixed ensemble

| Ket $|\psi_1\rangle = (0 - i \frac{1}{2})/\sqrt{2}$ | $|\psi_2\rangle = 1$ |
|---|---|
| Prob 3/4 | 1/4 |

Density Operators

- We now have a more powerful way of representing quantum states.
- Density operators are absolutely crucial when discussing and dealing with quantum noise and quantum decoherence.

Quantum Mechanics from the Two Perspectives

| Kets $|\psi\rangle$ | Density Ops $\rho$ |
|---|---|
| $i \hbar \frac{\partial |\psi\rangle}{\partial t} = H |\psi\rangle$ | $i \hbar \frac{\partial \rho}{\partial t} = [H, \rho]$ |
| $|\psi_0\rangle \mapsto U |\psi_0\rangle$ | $\rho_0 \mapsto U \rho_0 U^+ $ |
| $A = |\psi\rangle \langle A| \psi\rangle$ | $A = \text{tr}(A \rho)$ |


Quantum Computation and Information, Samuel J. Lomonaco, Jr. and Howard E. Brandt (editors), AMS CONM/305, (2002).
Measurement Revisited

\[ \psi = \sum_j \phi_j \]

where

\[ O = \sum_j \lambda_j P_j \]

Spectral Decomposition

\[ |\psi_j\rangle = \frac{P_j |\psi\rangle}{\sqrt{|P_j |\psi\rangle}} \]