Algebraic K-theory of group rings and the cyclotomic trace map

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Algebraic K-theory of group rings and the cyclotomic trace map

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Conjecture *If G is a torsion-free group, then $Wh(G) = 0$.*

- G is a discrete group
- the Whitehead group $Wh(G)$ is defined as

$$Wh(G) = K_1(\mathbb{Z}[G]) \,/\, \langle\, (\pm g) \mid g \in G \,\rangle$$

$$K_1(R) = \left(\bigcup_{k=1}^{\infty} GL_k(R) \right)_{ab} = GL(R)/E(R)$$

- s-Cobordism Theorem (Smale, Barden, Mazur, Stallings)
  $Wh(G) \cong \{$isomorphism classes of h-cobordisms over $M\}$
  if M is a closed manifold with $\pi_1(M) \cong G$ and $\dim(M) \geq 5$.
- $Wh(1) = 0$ implies the Poincaré Conjecture for S^n, $n \geq 5$.

Conjecture *If G is a torsion-free group, then* $Wh(G) = 0$.

Still open. No counterexamples. True if G is:

- free abelian (Bass-Heller-Swan) or free (Stallings)
- a classical knot or link group (Waldhausen)
- π_1(flat or negatively curved closed Riemannian manifold) (Farrell-Hsiang, Farrell-Jones)
- hyperbolic or CAT(0) (Bartels-Lück-Reich, Bartels-Lück)
- etc., etc., etc.

Favorite open case: Thompson's group F.

However, if G has torsion, then usually $Wh(G) \neq 0$.

- if $G = C$ is a finite cyclic group of order $c \notin \{1, 2, 3, 4, 6\}$ then $Wh(C) \neq 0$, even $Wh(C) \underset{\mathbb{Z}}{\otimes} \mathbb{Q} \neq 0$.

Theorem (Lück-Reich-Rognes-V.)

If $H_1(BZ_GC;\mathbb{Z})$ and $H_2(BZ_GC;\mathbb{Z})$ are finitely generated for every finite cyclic subgroup C of G, then there is an injective map

$$\operatorname*{colim}_{\text{finite } H\leq G} Wh(H) \underset{\mathbb{Z}}{\otimes} \mathbb{Q} \longrightarrow Wh(G) \underset{\mathbb{Z}}{\otimes} \mathbb{Q}\,.$$

$Z_GC =$ centralizer of C in G, colimit taken over the orbit category.

Corollary (Geoghegan-V.)

For Thompson's group T of orientation-preserving, dyadic, PL-homeomorphisms of the circle S^1, there is an injective map

$$\operatorname*{colim}_{c\in\mathbb{N}} Wh(C_c) \underset{\mathbb{Z}}{\otimes} \mathbb{Q} \longrightarrow Wh(T) \underset{\mathbb{Z}}{\otimes} \mathbb{Q}\,.$$

In particular, $\dim_{\mathbb{Q}} Wh(T) \underset{\mathbb{Z}}{\otimes} \mathbb{Q} = \infty\,.$

Loday assembly

$$H_1\big(BG; K(\mathbb{Z})\big) \xrightarrow{\alpha_L} K_1(\mathbb{Z}[G])$$

$$H_1\big(BG; K(\mathbb{Z})\big) \overset{\cong}{\longleftrightarrow} H_0\big(BG; K_1(\mathbb{Z})\big) \oplus H_1\big(BG; K_0(\mathbb{Z})\big) \overset{\cong}{\longleftrightarrow} H_0\big(BG; \{\pm 1\}\big) \oplus H_1\big(BG; \mathbb{Z}\big) \overset{\cong}{\longleftrightarrow} \{\pm 1\} \oplus G_{ab}$$

$$\{\pm 1\} \oplus G_{ab} \longrightarrow K_1(\mathbb{Z}[G]) \longrightarrow Wh(G) \longrightarrow 0$$

$$\mathrm{coker}\Big(\alpha_L :\ H_1\big(BG; K(\mathbb{Z})\big) \longrightarrow K_1(\mathbb{Z}[G])\Big) = Wh(G)$$

$$H_n\big(BG; K(\mathbb{Z})\big) \xrightarrow[\alpha_L]{\text{Loday assembly}} K_n(\mathbb{Z}[G])$$

Remark $\quad \operatorname{coker}\Big(\alpha_L : H_1\big(BG; K(\mathbb{Z})\big) \longrightarrow K_1(\mathbb{Z}[G])\Big) = Wh(G)$

- Hsiang Conjecture implies that $Wh(\mathrm{G}) = 0$
- α_L is usually not surjective if G has torsion

Conjecture (Hsiang)

If G is torsion-free, then the Loday assembly map α_L is an iso.

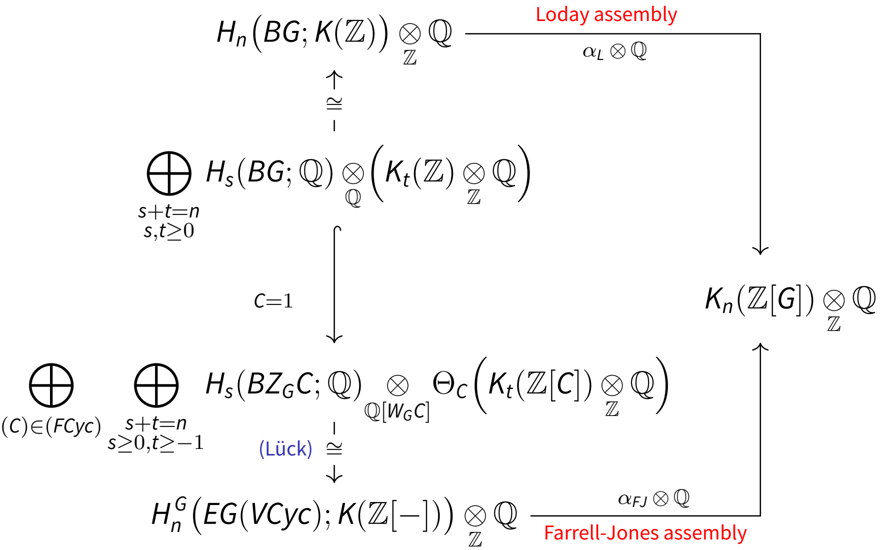
$$H_n\big(BG; K(\mathbb{Z})\big) \xrightarrow[\alpha_L]{\text{Loday assembly}} K_n(\mathbb{Z}[G])$$

iso if G is torsion-free: $H_n\big(BG; K(\mathbb{Z})\big) \to H_n^G\big(EG(VCyc); K(\mathbb{Z}[-])\big)$

$$H_n^G\big(EG(VCyc); K(\mathbb{Z}[-])\big) \xrightarrow[\text{Farrell-Jones assembly}]{\alpha_{FJ}} K_n(\mathbb{Z}[G])$$

Conjecture (Farrell-Jones)

For any group G the Farrell-Jones assembly map α_{FJ} is an iso.



Conjecture (Farrell-Jones)

For any group G the Farrell-Jones assembly map α_{FJ} is an iso.

Theorem (Lück-Reich-Rognes-V.)

The rationalized Farrell-Jones assembly map $(\alpha_{FJ} \otimes \mathbb{Q})|_{t \geq 0}$ *is injective if for all finite cyclic subgroups C of G:*

- [A] $H_s(BZ_GC; \mathbb{Z})$ *is finitely generated for each* $s \geq 0$*;*
- [B] *the map* $K_t(\mathbb{Z}[\zeta_c]) \longrightarrow \prod_{p \text{ prime}} K_t\Big(\mathbb{Z}_p \underset{\mathbb{Z}}{\otimes} \mathbb{Z}[\zeta_c]; \mathbb{Z}_p\Big)$ *is* $\mathbb{Q}$*-injective for each* $t \geq 0$*, where* $c = \#C$*.*

This generalizes (and reproves) the seminal:

Theorem (Bökstedt-Hsiang-Madsen)

The rationalized Loday assembly map $\alpha_L \otimes \mathbb{Q}$ *is injective if:*

- [A] $H_s(BG; \mathbb{Z})$ *is finitely generated for each* $s \geq 0$*.*

Theorem (Lück-Reich-Rognes-V.)

The rationalized Farrell-Jones assembly map $(\alpha_{FJ} \otimes \mathbb{Q})|_{t \geq 0}$ *is injective if for all finite cyclic subgroups C of G:*

[A] $H_s(BZ_GC;\mathbb{Z})$ *is finitely generated for each* $s \geq 0$*;*

[B] *the map* $K_t(\mathbb{Z}[\zeta_c]) \longrightarrow \prod_{p \text{ prime}} K_t\Big(\mathbb{Z}_p \underset{\mathbb{Z}}{\otimes} \mathbb{Z}[\zeta_c];\mathbb{Z}_p\Big)$ *is* $\mathbb{Q}$*-injective for each* $t \geq 0$*, where* $c = \#C$*.*

Assumption [A] is satisfied if:

- G has a universal space for proper actions $\underline{E}G = EG(Fin)$ which is of finite type, e.g., for hyperbolic groups, CAT(0) groups, arithmetic groups, mapping class groups, outer automorphism groups of free groups $Out(F_n)$, ...
- G is Thompson's group T (Geoghegan-V.)

Theorem (Lück-Reich-Rognes-V.)

The rationalized Farrell-Jones assembly map $(\alpha_{FJ} \otimes \mathbb{Q})|_{t \geq 0}$ *is injective if for all finite cyclic subgroups C of G:*

[A] $H_s(BZ_GC; \mathbb{Z})$ *is finitely generated for each* $s \geq 0$*;*

[B] *the map* $K_t(\mathbb{Z}[\zeta_c]) \longrightarrow \prod_{p \text{ prime}} K_t\Big(\mathbb{Z}_p \underset{\mathbb{Z}}{\otimes} \mathbb{Z}[\zeta_c]; \mathbb{Z}_p\Big)$ *is* $\mathbb{Q}$*-injective for each* $t \geq 0$*, where* $c = \#C$*.*

Assumption [B] is satisfied if:

- $c = 1$ and t is arbitrary
- c is arbitrary and $t = 0$ or 1
- c is fixed, for almost all $t \geq 0$
- the Leopoldt-Schneider Conjecture is true for $\mathbb{Q}(\zeta_c)$

Assumption [B] is conjecturally always satisfied.

Theorem (Lück-Reich-Rognes-V.)

The rationalized Farrell-Jones assembly map $(\alpha_{FJ} \otimes \mathbb{Q})|_{t \geq 0}$ *is injective if for all finite cyclic subgroups* C *of* G*:*

[A] $H_s(BZ_G C; \mathbb{Z})$ *is finitely generated for each* $s \geq 0$*;*

[B] *the map* $K_t(\mathbb{Z}[\zeta_c]) \longrightarrow \prod_{p \text{ prime}} K_t\Big(\mathbb{Z}_p \underset{\mathbb{Z}}{\otimes} \mathbb{Z}[\zeta_c]; \mathbb{Z}_p\Big)$

is $\mathbb{Q}$*-injective for each* $t \geq 0$*, where* $c = \#C$*.*

Corollary

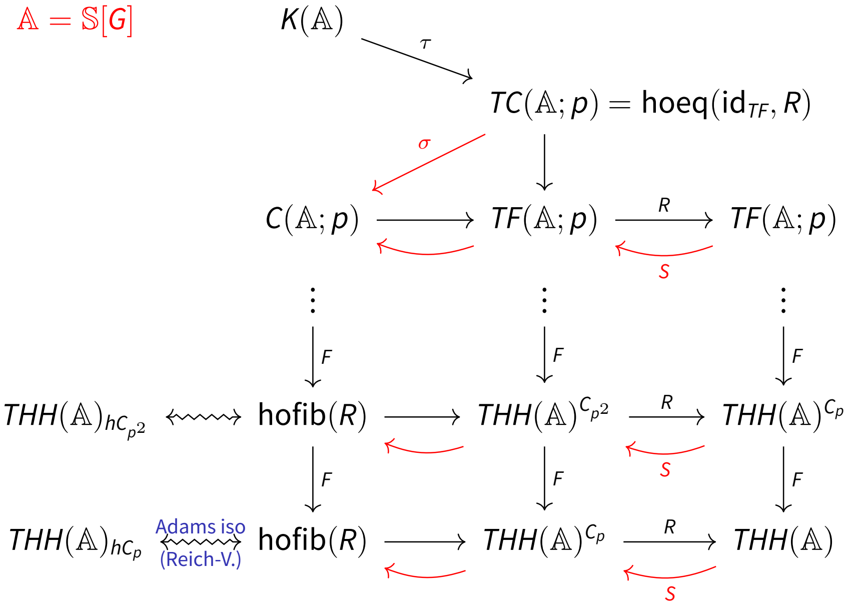
If G *has a finite universal space for proper actions* $EG(Fin)$*, then there exists an* $N > 0$ *such that the rationalized Farrell-Jones assembly map* $\alpha_{FJ} \otimes \mathbb{Q}$ *is injective in all dimensions* $n \geq N$*.*

$$
\begin{array}{ccc}
H_n^G\big(EG(VCyc);K(\mathbb{Z}[-])\big) & \xrightarrow{\alpha_{FJ}} & K_n(\mathbb{Z}[G]) \\
\mathbb{Q}\text{-iso}\uparrow & & \| \\
H_n^G\big(EG(Fin);K(\mathbb{Z}[-])\big) & \longrightarrow & K_n(\mathbb{Z}[G]) \\
\mathbb{Q}\text{-iso}\uparrow \ell_\% & & \mathbb{Q}\text{-iso}\uparrow \ell \\
H_n^G\big(EG(Fin);K(\mathbb{S}[-])\big) & \longrightarrow & K_n(\mathbb{S}[G]) \\
\downarrow \tau_\% & & \downarrow \tau \\
H_n^G\Big(EG(Fin);\prod TC(\mathbb{S}[-];p)\Big) & \longrightarrow & \pi_n\Big(\prod TC(\mathbb{S}[G];p)\Big) \\
\downarrow \sigma_\% & & \downarrow \sigma \\
H_n^G\Big(EG(Fin);THH(\mathbb{S}[-])\times\prod C(\mathbb{S}[-];p)\Big) & \xrightarrow[\alpha]{} & \pi_n\Big(THH(\mathbb{S}[G])\times\prod C(\mathbb{S}[G];p)\Big)
\end{array}
$$

Want to show that $\alpha_{FJ}\otimes\mathbb{Q}$ is injective.

Detection Theorem [B] implies that $(\sigma_\%\circ\tau_\%)\otimes\mathbb{Q}$ is injective.

Splitting Theorem [A] implies that $\alpha\otimes\mathbb{Q}$ is injective. QED



$\mathbb{A} = \mathbb{S}[G]$
$K(\mathbb{A})$
τ
$TC(\mathbb{A};p) = \mathrm{hoeq}(\mathrm{id}_{TF}, R)$
σ
$C(\mathbb{A};p)$
$TF(\mathbb{A};p)$
R
$TF(\mathbb{A};p)$
S
F
F
F
$THH(\mathbb{A})_{hC_{p^2}}$
$\mathrm{hofib}(R)$
$THH(\mathbb{A})^{C_{p^2}}$
R
$THH(\mathbb{A})^{C_p}$
S
F
F
F
$THH(\mathbb{A})_{hC_p}$
Adams iso
(Reich-V.)
$\mathrm{hofib}(R)$
$THH(\mathbb{A})^{C_p}$
R
$THH(\mathbb{A})$
S

Thank You!

Happy Birthday, Don!