Math 461: Topology, Fall 2009 — Final Exam, December 16

Name: ....

I	II	III	IV	V	VI	TOTAL
39	14	11	11	11	14	100

I. True or false? Please circle your answers.
(1.5 points for each correct answer, but **be careful**: 1 point will be subtracted for each wrong answer!)

1] *Math 461: Topology* is by far the best class you have ever taken. .... TRUE | FALSE

2] If $f\colon X \to Y$ is a function and X is countable, then $f(X)$ is countable. .... TRUE | FALSE

3] If $f\colon X \to Y$ is a function and Y is countable, then $f(X)$ is countable. .... TRUE | FALSE

4] If $f\colon X \to Y$ is a function and Y is countable, then $f^{-1}(Y)$ is countable. ...... TRUE | FALSE

5] If X is a discrete topological space,
then any subspace of X is discrete. .... TRUE | FALSE

6] If X is a topological space which is not discrete (i.e., not all subsets are open),
then **no** subspace of X is discrete. .... TRUE | FALSE

7] If X is a disconnected topological space,
then any subspace of X is disconnected. .... TRUE | FALSE

8] If X is a path-connected topological space,
then any subspace of X is path-connected. .... TRUE | FALSE

9] If X is a compact topological space,
then any **closed** subspace of X is compact. .... TRUE | FALSE

10] If X is a Hausdorff topological space,
then any subspace of X is Hausdorff. .... TRUE | FALSE

11] If X is a second-countable topological space,
then any subspace of X is second-countable. .... TRUE | FALSE

12] If X is a second-countable topological space,
then any **basis** for the topology of X is countable. .... TRUE | FALSE

True or false? (Continued.)

13] If A and B are connected subspaces of a topological space X and $A \cap B \neq \emptyset$, then $A \cup B$ is connected. .... TRUE | FALSE

14] If A and B are connected subspaces of a topological space X and $A \cap B \neq \emptyset$, then $A \cap B$ is connected. .... TRUE | FALSE

15] If $f\colon X \to Y$ is a **continuous** function between topological spaces X and Y, then for every open subset U of X, $f(U)$ is open in Y. .... TRUE | FALSE

16] If $f\colon X \to Y$ is a **homeomorphism** between topological spaces X and Y, then for every open subset U of X, $f(U)$ is open in Y. .... TRUE | FALSE

17] If $f\colon X \to Y$ is a **bijective** function between topological spaces X and Y, and for every open subset U of X, $f(U)$ is open in Y, then f is a homeomorphism. TRUE | FALSE

18] If X is a Hausdorff space, Y is a compact space, and $f\colon X \to Y$ is a continuous and bijective function, then f is a homeomorphism. .... TRUE | FALSE

19] If X and Y are both **compact metric** spaces, and $f\colon X \to Y$ is a continuous and bijective function, then f is a homeomorphism. .... TRUE | FALSE

20] $\mathbb{R}$ and $\mathbb{R}^2$ with the standard topologies are homeomorphic. .... TRUE | FALSE

21] $\mathbb{Z}$ and $\mathbb{Z}^2$ with the discrete topologies are homeomorphic. .... TRUE | FALSE

22] If $f\colon X \to Y$ is a continuous function between topological spaces X and Y, and X is connected and compact, then $f(X)$ is connected and compact. .... TRUE | FALSE

23] If $f\colon X \to Y$ is a continuous function between topological spaces X and Y, and X is separable, then $f(X)$ is separable. .... TRUE | FALSE

24] If $f\colon X \to Y$ is a continuous function between topological spaces X and Y, and X is Hausdorff, then $f(X)$ is Hausdorff. .... TRUE | FALSE

25] If $X = \mathbb{R}$ is given the cofinite (also known as finite complement) topology, then the function $f\colon X \to X$, $f(x) = \sin(x)$, is continuous. .... TRUE | FALSE

26] If $X = \mathbb{R}$ is given the cofinite (also known as finite complement) topology, then the function $f\colon X \to X$, $f(x) = x^2$, is continuous. .... TRUE | FALSE

39 TOTAL POINTS

II. Fill in the blanks in the following theorem.

Theorem. Let X and Y be topological spaces, and let $f\colon X \to Y$ be a function. Then the following conditions are equivalent:

(i) f is continuous, i.e., for every open subset U of Y, $f^{-1}(U)$ is ....;

(ii) for every closed subset C of Y, $f^{-1}(C)$ is ....;

(iii) for every subset A of X, one has $f(\bar{A})$ ....;

(iv) for every subset B of Y, one has $f^{-1}(\bar{B})$...;

(v) for every point $x \in X$ and every neighborhood V of $f(x)$ in Y, there is ....

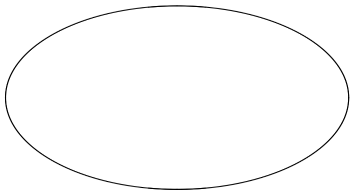
....

....

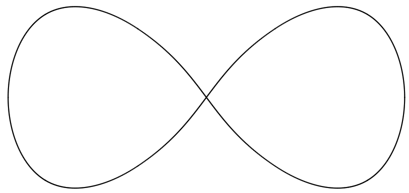
Prove **exactly two** implications of your choice from this theorem.

14 TOTAL POINTS

III. Consider the following two subspaces of $\mathbb{R}^2$ with the standard topology.



X



Y

Are X and Y homeomorphic? Justify your answer carefully.

11 TOTAL POINTS

IV. Complete the following definition.

Definition. If X is a topological space and A is a subset of X, then the *closure* of A in X is

$$\overline{A} = \left\{ x \in X \;\middle|\; \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \right\}.$$

- If $X = \mathbb{R}^2$ with the **standard** topology and $A = \{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1 \text{ and } x \neq 0 \}$, then what is $\overline{A}$?

- Now let X be a set and let A be a non-empty subset of X.

  The possible answers for the three following questions are as follows:

  (1) $\overline{A} = A$.

  (2) $\overline{A} = X$.

  (3) $\overline{A} = \begin{cases} A & \text{if } A \text{ is finite,} \\ X & \text{if } A \text{ is infinite.} \end{cases}$

  Write the number corresponding to the correct answer in each of the boxes below.

  - If X has the **indiscrete** topology and $\emptyset \neq A \subset X$, then .... ☐
  - If X has the **discrete** topology and $\emptyset \neq A \subset X$, then .... ☐
  - If X has the **cofinite** (also known as finite complement) topology and $\emptyset \neq A \subset X$, then ☐

11 TOTAL POINTS

V. Complete the following two definitions, and then write the precise **statement (without proof!)** of **either** the intermediate value theorem **or** the extreme value theorem.

Definition. A topological space X is *disconnected* if and only if ....

....

....

....

....

Definition. A topological space X is *compact* if and only if ....

....

....

....

....

.... Value Theorem. Let X be a topological space, and let $f\colon X \to \mathbb{R}$ be a function.

Assume that X is ....

and that f is ....

Then ....

....

....

....

....

11 TOTAL POINTS

VI. Solve **only one** of the following two problems.

A] Recall that S^0 denotes the topological space with only two points $\{+1, -1\}$ and the discrete topology. Prove that a topological space X is disconnected if and only if there exists a continuous and surjective function $f\colon X \to S^0$.

B] Recall the following result that we proved in class.

Lemma. If C is a compact subset of a Hausdorff space X and $x \in X - C$, then there exist open subsets U and V of X such that $C \subset U$, $x \in V$, and $U \cap V = \emptyset$.

Now let X be a Hausdorff space, and let C and D be compact subsets of X such that $C \cap D = \emptyset$. Prove that there exist open subsets U and V of X such that $C \subset U$, $D \subset V$, and $U \cap V = \emptyset$.

14 TOTAL POINTS