Let $\mathbb{Z}$ denote the set of all integers, and let $\mathbb{P} = \{2, 3, 5, 7, 11, 13, 17, \ldots\}$ denote the set of all prime numbers. In this homework assignment you will be guided through a topological proof of the well-known fact that there are infinitely many primes, i.e., that $\mathbb{P}$ is an infinite set.

Definition. For any $c, d \in \mathbb{Z}$ with $d > 0$, let $AP_{c,d} = \{\, c + nd \mid n \in \mathbb{Z} \,\}$ be the infinite arithmetic progression with initial term c and common difference d. Define $\mathcal{B} = \{\, AP_{c,d} \mid c, d \in \mathbb{Z},\ d > 0 \,\}$.

Prove the following three lemmas.

Lemma 1. *The set $\mathcal{B}$ is a basis for a topology on $\mathbb{Z}$.*

Let's consider $\mathbb{Z}$ together with the topology generated by $\mathcal{B}$.

Lemma 2. *Every non-empty open set of $\mathbb{Z}$ is infinite.*

Lemma 3. *For any $c, d \in \mathbb{Z}$ with $d > 0$, the set $AP_{c,d}$ is closed.*

Now consider $X = \mathbb{Z} - \left(\bigcup_{p \in \mathbb{P}} AP_{0,p} \right)$. What exactly is the set X? Is X open?

From all this, deduce the following statement.

Theorem 4. *There are infinitely many primes.*