Derivatives of Schubert polynomials and proof of a determinant conjecture of Stanley

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Based on joint work with Zachary Hamaker, Oliver Pechenik, and David E Speyer.

Hamaker, Z., Pechenik, O., Speyer, D. E., and Weigandt, A. (2020). Derivatives of Schubert polynomials and proof of a determinant conjecture of Stanley. *Algebraic Combinatorics*, 3(2):301–307

Schubert Polynomials

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Schubert Polynomials were introduced by Lascoux and Schützenberger (1982) to study the cohomology of the **complete flag variety**.

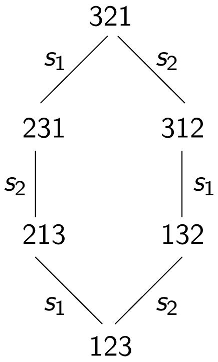
- Start with the **longest** permutation in $\mathcal{S}_n$

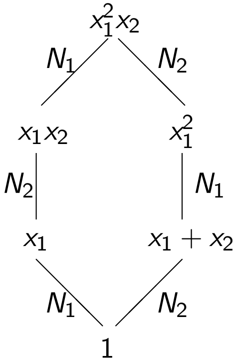
$$w_0 = n\, n-1 \dots 1 \qquad \mathfrak{S}_{w_0} := x_1^{n-1} x_2^{n-2} \cdots x_{n-1}$$

- The rest are defined recursively by **Newton's divided difference operators:**

$$N_i f := \frac{f - s_i \cdot f}{x_i - x_{i+1}} \quad \text{and} \quad N_i \mathfrak{S}_w := \begin{cases} \mathfrak{S}_{ws_i} & \text{if } w(i) > w(i+1) \\ 0 & \text{otherwise.} \end{cases}$$

Schubert Polynomials for $\mathcal{S}_3$





Zach's Question: What happens when you take "the" derivative of a Schubert polynomial?

Example:

$$\mathfrak{S}_{2413} = x_1^2 x_2 + x_1 x_2^2.$$

$$\frac{\partial}{\partial x_1}(\mathfrak{S}_{2143}) = 2x_1x_2 + x_2^2 \qquad \frac{\partial}{\partial x_2}(\mathfrak{S}_{2143}) = x_1^2 + 2x_1x_2$$

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$$\frac{\partial}{\partial x_1}(\mathfrak{S}_{2413}) = \mathfrak{S}_{1423} - \mathfrak{S}_{3124} + \mathfrak{S}_{2314} \qquad \frac{\partial}{\partial x_2}(\mathfrak{S}_{2413}) = \mathfrak{S}_{3124} + 2\mathfrak{S}_{2314}$$

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$$\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right)(\mathfrak{S}_{2413}) = \mathfrak{S}_{1423} + 3\mathfrak{S}_{2314}.$$

Proposition (Hamaker-Pechenik-Speyer-Weigandt, 2020)

Let $\nabla = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}$. *Then*

$$\nabla(\mathfrak{S}_w) = \sum_{s_k w < w} k \mathfrak{S}_{s_k w}.$$

Proof Sketch

One checks that $\nabla N_i = N_i \nabla$ for all $i \in [n-1]$.

Verify directly that $\nabla(\mathfrak{S}_{w_0}) = \sum_{k=1}^{n-1} k \mathfrak{S}_{s_k w_0}$.

We can write $w s_{a_1} s_{a_2} \cdots s_{a_m} = w_0$ where $(a_1, \ldots, a_m)$ is a **reduced word**.

In particular, $w = w_0 s_{a_m} \cdots s_{a_1}$. Therefore,

$$\mathfrak{S}_w = N_{a_1} \cdots N_{a_m} \mathfrak{S}_{w_0}.$$

We have

$$N_{a_1} \cdots N_{a_m} \mathfrak{S}_{s_k w_0} = \begin{cases} \mathfrak{S}_{s_k w} & \text{if} \quad s_k w < w \quad \text{and} \\ 0 & \text{otherwise.} \end{cases}$$

Proof Sketch (Continued)

Therefore,

$$\nabla(\mathfrak{S}_w) = \nabla N_{a_1} \cdots N_{a_m} \mathfrak{S}_{w_0}$$

$$= N_{a_1} \cdots N_{a_m} \nabla(\mathfrak{S}_{w_0})$$

$$= N_{a_1} \cdots N_{a_m} \sum_{k=1}^{n-1} k \mathfrak{S}_{s_k w_0}$$

$$= \sum_{s_k w < w} k \mathfrak{S}_{s_k w}.$$

□

Macdonald's Identity

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Theorem (Macdonald, 1991)

$$\frac{1}{\ell(w)!}\sum_{a\in R(w)} a_1a_2\cdots a_{\ell(w)} = \mathfrak{S}_w(1,1,\ldots,1).$$

Macdonald, I. G. (1991). *Notes on Schubert polynomials*, volume 6. Publications du LACIM, Université du Québec à Montréal

Fomin, S. and Kirillov, A. N. (1997). Reduced words and plane partitions. *J. Algebraic Combin.*, 6(4):311–319

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Example

$\mathfrak{S}_{2413} = x_1^2 x_2 + x_1 x_2^2$ and $R(2413) = \{312, 132\}$.

We verify

$$\frac{1}{3!}(3\cdot 1\cdot 2 + 1\cdot 3\cdot 2) = 2.$$

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A (Short) Proof of Macdonald's Identity

If m is a (monic) monomial of degree k, then $\nabla^k(m) = k!$.

$\mathfrak{S}_w$ is homogeneous of degree $\ell(w)$ and therefore

$$\nabla^{\ell(w)}(\mathfrak{S}_w) = \ell(w)!\mathfrak{S}_w(1, 1, \ldots, 1).$$

On the other hand, since $\nabla(\mathfrak{S}_w) = \sum_{s_k w < w} k\mathfrak{S}_{s_k w}$,

$$\nabla^{\ell(w)}(\mathfrak{S}_w) = \sum_{a \in R(w)} a_1 a_2 \cdots a_{\ell(w)}.$$

$$\ell(w)!\mathfrak{S}_w(1, 1, \ldots, 1) = \sum_{a \in R(w)} a_1 a_2 \cdots a_{\ell(w)}.$$

□

The Sperner Property

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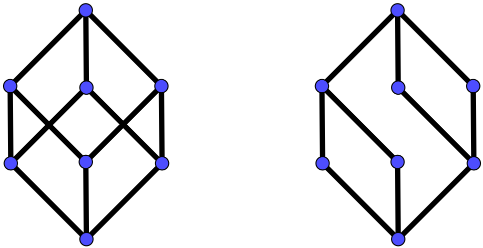
A graded poset is **Sperner** if it has some rank level which is an antichain of the maximum possible size.



Sperner   Not Sperner

Strongly Sperner Posets

A graded poset $\mathcal{P} = \bigsqcup_{i=0}^{m} \mathcal{P}_i$ is k-**Sperner** if

$\max\{|\mathcal{S}| : \mathcal{S} \subseteq \mathcal{P}$ so that $\mathcal{S}$ does not contain any chain of size $k+1\}$

equals

$$\max\{|\mathcal{P}_{i_1} \cup \cdots \cup \mathcal{P}_{i_k}| : 0 \leq i_1 < i_2 < \cdots < i_k \leq m\}.$$

If $\mathcal{P}$ is k-Sperner for all $k \geq 1$ then we say $\mathcal{P}$ is **strongly Sperner**.

Theorem (Stanley, 1980)

Suppose there is an order raising operator $U : \mathbb{Q}\mathcal{P} \to \mathbb{Q}\mathcal{P}$ *such that if* $0 \leq k < \frac{m}{2}$ *then*

$$U^{m-2k} : \mathbb{Q}\mathcal{P}_k \to \mathbb{Q}\mathcal{P}_{m-k}$$

is a bijection. Then $\mathcal{P}$ *is strongly Sperner.*

Stanley, R. P. (1980). Weyl groups, the hard Lefschetz theorem, and the Sperner property. *SIAM J. Algebraic Discrete Methods*, 1(2):168–184

The Weak Order on the Symmetric Group

In 2017, Stanley conjectured that the **weak order on the symmetric group is strongly Sperner** and gave a candidate for an order raising operator defined by

$$U(w) = \sum_{s_i w > w} i \cdot s_i w.$$

Stanley also conjectured an explicit formula for the determinant of the operator

$$U^{\binom{n}{2}-2k} : \mathbb{Q}\mathcal{S}_n(k) \to \mathbb{Q}\mathcal{S}_n\left(\binom{n}{2} - k\right)$$

which is nonzero.

In 2018, Christian Gaetz and Yibo Gao proved that the weak order on $\mathcal{S}_n$ is strongy Sperner.

Gaetz, C. and Gao, Y. (2019). A combinatorial sl2-action and the Sperner property for the weak order. *Proceedings of the American Mathematical Society*

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Theorem (Hamaker-Pechenik-Speyer-Weigandt)

For $k \leq \binom{n}{2} - k$, *we have* $\det U^{\binom{n}{2}-2k} =$

$$\pm \prod_{i=0}^{k} \left((k-i+1)(k-i+2)\cdots\left(\binom{n}{2}-k-i\right)\right)^{|\mathcal{S}_n(i)|-|\mathcal{S}_n(i-1)|}.$$

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Idea of the Proof:

Let $\mathcal{W} = \{x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} : 0 \leq a_i \leq n - i\}$.

Schubert polynomials are a basis for the vector space $\mathbb{Q}\mathcal{W}$.

The base change matrix from $\{\mathfrak{S}_w : w \in \mathcal{S}_n\}$ to $\mathcal{W}$ has determinant ± 1.

The action of

$$U : \mathbb{Q}\mathcal{S}_n \to \mathbb{Q}\mathcal{S}_n$$

essentially equivalent to

$$\nabla : \mathbb{Q}\mathcal{W} \to \mathbb{Q}\mathcal{W}.$$

Computing $\det(U^{\binom{n}{2}-2k})$ amounts to computing

$$\det\left(\mathbb{Q}\mathcal{W}_{\binom{n}{2}-k} \xrightarrow{\nabla^{\binom{n}{2}-2k}} \mathbb{Q}\mathcal{W}_k\right).$$

We can consider the action of ∇ (and hence U) with respect to the monomial basis!

We interpret ∇ as an $\mathfrak{sl}_2$ action and then use tricks from representation theory.

$\square$

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Thank you!