Hessenberg Schubert Polynomials

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March 20, 2021

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Supported by SWIM-MSRI and NSF-DMS-1800773

Hessenberg Schubert Polynomials

I. (Regular Nilpotent) Hessenberg varieties

II. A recursive construction of the cohomology of regular nilpotent Hessenberg varieties

III. Hessenberg Schubert polynomials

I. Hessenberg varieties: basic properties

<u>Flag variety</u> $GL_n(\mathbb{C})/B$ ← Borel, upper Δ matrices

$$\text{Flag} \quad gB = V_\bullet = \{ V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_{n-1} \subsetneq \mathbb{C}^n \}$$

<u>Linear operator</u> $X : \mathbb{C}^n \to \mathbb{C}^n$

eg $X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ (single Jordan block)

I. Hessenberg varieties: basic properties

First example: Springer fiber of X

"eigenflags" of X

$$\{\text{flags } V_\bullet \text{ with } XV_i \subseteq V_i \ \forall i\}$$

$$= \{\text{flags } gB \text{ with } g^{-1}Xg \text{ upper } \Delta\}$$

I. Hessenberg varieties: basic properties

<u>Definition</u>: Hessenberg variety of X, h

where $h: \{1, 2, \dots, n\} \to \{1, 2, \dots, n\}$
satisfies $h(i) \geq i$, $h(i) \geq h(i-1)\ \forall i$

$$\mathcal{H}(X,h) = \{\text{flags } V_\bullet \text{ with } XV_i \subseteq V_{h(i)}\ \forall i\}$$

$$= \{\text{flags } gB \text{ with } g^{-1}Xg \text{ of shape } H\}$$

$$H = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & 0 & * \end{pmatrix}$$

i^{th} column has $h(i)$ nonzero entries

I. Hessenberg varieties: basic properties

Remarks for Lie theorists:

1) The definition generalizes to all Lie types

linear subspace

$\{ \text{Flags } gB : \ Ad(g^{-1})(X) \in H \}$ of $\mathfrak{g}$

2) H is not parabolic: if $h(i) = i+1 \ \forall i$ then H contains all negative simple root spaces

I. Hessenberg varieties: basic properties

Questions prior researchers considered:

1) Special families of X

- Regular semisimple: $X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$
- Regular nilpotent: X has 1 Jordan block
- Nilpotent: eigenvalue is 0

I. Hessenberg varieties: basic properties

Questions prior researchers considered:

2) Special families of h

- Springer fibers: $h(i) = i \;\forall i$
- Parabolic h:

$$H = \begin{array}{|c|c|c|}\hline * & * & * \\ \hline 0 & * & * \\ \hline 0 & 0 & * \\ \hline \end{array}$$ block upper triangular

- Peterson varieties

$$h(i) = i+1 \;\forall i, \quad X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

II. Recursive construction of $H^*(\mathcal{H}(X,h))$ when X is regular nilpotent

Motivation: Sommers-T proved

reg nilp

Poincaré Polynomial of $\mathcal{H}(X,h) =$

$$\prod_{i=1}^{n} \left(1 + t + t^2 + \cdots + t^{h(i)-i}\right)$$

" like an iterated tower of $\mathbb{P}^{h(i)-i}$ bundles "

II. Recursive construction of $H^*(\mathcal{H}(X,h))$

EG: $h(i) = i \quad \forall i$ $\qquad \prod_{i=1}^{n} (1) = 1$

reg nilp Springer fiber

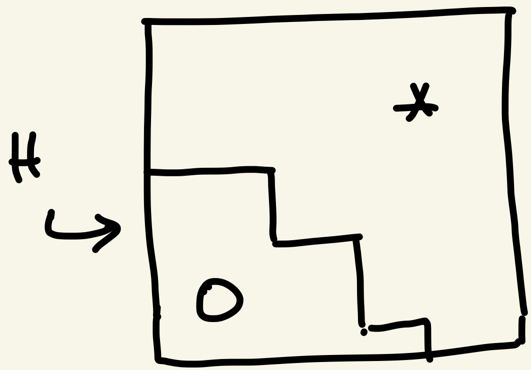
EG: $h(i) = i+1 \quad \forall i$ $\qquad \prod_{i=1}^{n-1} (1+t)$

Peterson variety

EG: $h(i) = n \quad \forall i$ $\qquad$ flag variety

$$\prod_{i=1}^{n-1} (1+t+t^2+\cdots+t^{n-i}) = \prod_{i=1}^{n-1} (1+t+\cdots+t^i)$$

II. Recursive construction of $H^*(\mathcal{H}e(X,h))$



$H^{(i)}$: erase row i & column i

<u>Observation:</u> $H^{(i)}$ is still a Hessenberg space / $h^{(i)}$ is still a Hessenberg function

.... but not actually $\mathbb{P}^d$-like ....

II. Recursive construction of $H^*(\mathcal{H}(X,h))$

<u>Theorem</u>: [Abe, Harada, Horiguchi, Masuda]

$$H^*(\mathcal{H}(X,h)) \cong \mathbb{C}[x_1,\dots,x_n] / I_h$$

where I_h is generated by explicit set of polynomials

$$f_{h(i),i} = \sum_{j=1}^{i} \prod_{k=i+1}^{h(i)} (x_j - x_k) x_j \quad \text{for } i=1,\dots,n$$

II. Recursive construction of $H^*(\mathcal{H}(X,h))$

We want generators for I_h that are closer to symmetric. Swap generators out one by one with

$$g_{h(i),i} = \sum_{j=1}^{i} \prod_{k=i+1}^{h(i)} (x_j - x_k) \boxed{}$$

differs from $f_{h(i),i}$ by factor here

II. Recursive construction of $H^*(\mathcal{H}(X,h))$

We get a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & R^h_n \arrow[r] & \cdots \arrow[r] & R^h_2 \arrow[r] & R^h_1 \cong H^*(\mathcal{H}(X,h)) \\
& & & \dfrac{\mathbb{C}[x_1,\dots,x_n]}{\langle g_{h(1),1}, f_{h(2),2}, f_{h(3),3}, \dots \rangle} \arrow[u] & \text{all } f\text{'s} \arrow[u] \\
& \dfrac{\mathbb{C}[x_1,\dots,x_n]}{\langle g_{h(1),1}, g_{h(2),2}, g_{h(3),3}, \dots, g_{h(n),n} \rangle} \arrow[uu, bend right] & & &
\end{tikzcd}$$

II. Recursive construction of $H^*(\mathcal{H}(X,h))$

We get a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{R}^h_n \arrow[r] \arrow[d, "\cong"] & \cdots \arrow[r] & \mathcal{R}^h_2 \arrow[r] \arrow[d] & \mathcal{R}^h_1 \cong H^*(\mathcal{H}(X,h)) \arrow[d] \\
& \mathcal{R}^{h^{(n)}}_{r_n} & & \mathcal{R}^{h^{(2)}}_{r_2} & \mathcal{R}^{h^{(1)}}_{r_1}
\end{tikzcd}$$

II. Recursive construction of $H^*(\mathcal{H}(X,h))$

We get a commutative diagram

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 & \mathcal{R}^{h(n)}_{r_n} & & \mathcal{R}^{h(2)}_{r_2} & \mathcal{R}^{h(1)}_{r_1}
\end{tikzcd}$$

$$\boxed{\begin{matrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ o & o & \times & \times & \times \\ o & o & o & \times & \times \end{matrix}}$$

$$r_i = \min\{ j : h(j) \geq i \}$$

II. Recursive construction of $H^*(\mathcal{H}(X,h))$

<u>Theorem</u> [Harada, Horiguchi, Murai, Precup, T]

We get a commutative diagram

$$
\begin{tikzcd}
0 \arrow[r] & \mathcal{R}^h_n \arrow[r] \arrow[d, "\cong"] & \cdots \arrow[r] & \mathcal{R}^h_2 \arrow[r] \arrow[d] & \mathcal{R}^h_1 \cong H^*(\mathcal{H}(X,h)) \arrow[d] \\
 & \mathcal{R}^{h(n)}_{r_n} & & \mathcal{R}^{h(2)}_{r_2} & \mathcal{R}^{h(1)}_{r_1}
\end{tikzcd}
$$

III. Hessenberg Schubert Polynomials

Consequences:

1) An alternate proof of the Sommers-T result on Poincaré polynomials

$$\prod_{i=1}^{n} \left(1 + t + t^2 + \dots + t^{h(i)-i}\right)$$

III. Hessenberg Schubert Polynomials

Consequences:

2) The monomials

$$\left\{ x_1^{e_1} x_2^{e_2} \cdots x_n^{e_n} : 0 \le e_i \le h(i)-i \right\}$$

form a basis for $H^*(\mathcal{H}(X,h))$

This proves a conjecture of Mbirika

III. Hessenberg Schubert Polynomials

Consequences:

3) It makes sense to talk about Hessenberg Schubert polynomials

$$\left[\,\overline{C_w \cap \mathcal{H}(X,h)}\,\right] = \sum_{\substack{e=(e_1,\dots,e_n) \\ \text{such that} \\ 0 \le e_i \le h(i)-i \ \forall i}} a_e \, x_1^{e_1} \cdots x_n^{e_n}$$

III. Hessenberg Schubert Polynomials

Consequences:

3) It makes sense to talk about Hessenberg Schubert polynomials

$$\left[\,\overline{C_w \cap \mathcal{H}(X,h)}\,\right] = \sum a_e \, x_1^{e_1} \cdots x_n^{e_n}$$

Question: What are these a_e?

Thank You!