Equivariant cohomology, Schubert calculus, and Edge labeled tableaux

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$H^*(X)$

Let $X = \mathrm{Gr}(k,n)$. The opposite Borel $\mathrm{B}_- \subset \mathrm{GL}_n$ acts on X with finitely many orbits. The **Schubert varieties** X_λ are closures of these orbits. X_λ admits a **Schubert class** $\sigma_\lambda(X)$ in $H^*(X)$.

Since these form a $\mathbb{Z}$-basis for $H^*(X)$,

$$\sigma_\lambda(X) \smile \sigma_\mu(X) = \sum_{\nu \subseteq k \times (n-k)} c_{\lambda,\mu}^{\nu} \sigma_\nu(X), \text{ where } c_{\lambda,\mu}^{\nu} \in \mathbb{Z}_{\geq 0}.$$

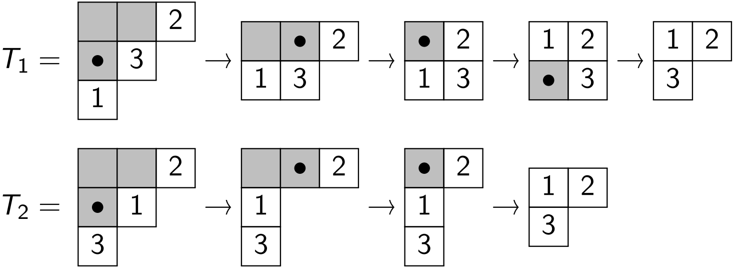
These $c_{\lambda,\mu}^{\nu}$ are the **Littlewood-Richardson coefficients**.

Structure coefficients for $H^*(X)$

Theorem

$$c^{\nu}_{\lambda,\mu} = \#\{T \in \mathrm{SYT}(\nu/\lambda) : \mathrm{Rect}(T) = S_\mu\}$$

Below are those $T \in \mathrm{SYT}((3,2,1)/(2,1))$ such that $\mathrm{Rect}(T) = S_{(2,1)}$, so $c^{(3,2,1)}_{(2,1),(2,1)} = 2$.



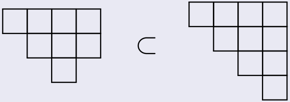
Lagrangian Grassmannian

Let $Z = \mathrm{LG}(n, 2n)$ be the **Lagrangian Grassmannian** of n-dimensional isotropic subspaces of $\mathbb{C}^{2n}$.

The **Schubert classes** $\sigma_\lambda(Z) \in H^*(Z)$ are indexed by strict partitions fitting inside the shifted staircase

$$\rho_n = (n, n-1, n-2, \ldots, 3, 2, 1).$$

Example: $\lambda = (4, 3, 1) \subset \rho_4$



Structure coefficients for $H^*(Z)$

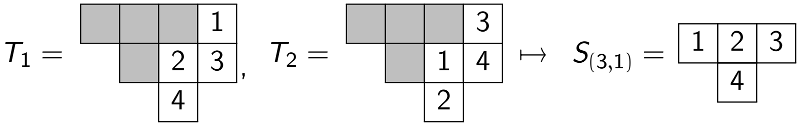
These $\sigma_\lambda(Z)$ form a $\mathbb{Z}$-linear basis of $H^*(Z)$

$$\sigma_\lambda(Z) \smile \sigma_\mu(Z) = \sum_\nu f_{\lambda,\mu}^{\nu} \sigma_\nu(Z), \text{ where } f_{\lambda,\mu}^{\nu} \in \mathbb{Z}_{\geq 0}.$$

Theorem [Worley (1977)]

$$f_{\lambda,\mu}^{\nu} = 2^{\ell(\nu)-\ell(\lambda)-\ell(\mu)} \cdot \#\{T \in \mathrm{SYT}(\nu/\lambda) : \mathrm{Rect}(T) = S_\mu\}.$$

Let $\lambda = \mu = (3,1), \nu = (4,3,1)$. Below are those $T \in \mathrm{SYT}(\nu/\lambda)$ such that $\mathrm{Rect}(T) = S_\mu$. Thus, $f_{\lambda,\mu}^{\nu} = 2^{3-2-2} \cdot 2 = 1$ since



Structure coefficients for $H_T^*(Z)$

Consider equivariant Schubert classes $\xi_\lambda(Z) \in H_T^*(Z)$. Then

$$\xi_\lambda(Z) \cdot \xi_\mu(Z) = \sum_{\nu \subseteq \rho_n} F_{\lambda,\mu}^{\nu} \xi_\nu(Z).$$

Theorem [Graham (2001)]

$$F_{\lambda,\mu}^{\nu} \in \mathbb{Z}_{\geq 0}[\gamma_1, \gamma_2, \ldots, \gamma_n],$$

where the γ_i are simple roots.

Problem

Determine a combinatorial rule to compute $F_{\lambda,\mu}^{\nu}$.

Anderson-Fulton ring

Anderson-Fulton introduced a ring Γ with $\mathbb{Z}[z]$-basis $\{\Omega_\lambda\}_{\lambda\subseteq\rho_n}$. Define structure coefficients by

$$\Omega_\lambda \cdot \Omega_\mu = \sum_{\nu\subseteq\rho_n} L^\nu_{\lambda,\mu}\Omega_\nu.$$

Problem

Determine a combinatorial rule to compute $L^\nu_{\lambda,\mu}$.

Anderson-Fulton connected Γ to the equivariant Schubert calculus of Z. That is,

$$F^\nu_{\lambda,\mu}(\alpha_1 \mapsto z, \alpha_2 \mapsto 0, \ldots, \alpha_n \mapsto 0) = L^\nu_{\lambda,\mu}.$$

Shifted Edge Labeled tableaux

For $\lambda, \mu, \nu \subseteq \rho_n$, let

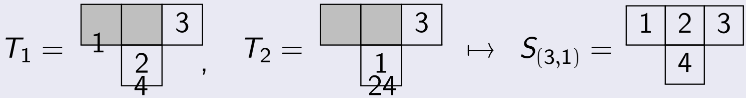
$$d^{\nu}_{\lambda,\mu} := \#\{T \in \mathrm{eqSYT}(\nu/\lambda, |\mu|) : \mathrm{eqRowRect}(T) = S_{\mu}\}.$$

Define $D^{\nu}_{\lambda,\mu} = 2^{L-\Delta} z^{\Delta} d^{\nu}_{\lambda,\mu}$, where

$$\Delta = |\lambda| + |\mu| - |\nu| \quad \text{and} \quad L = \ell(\lambda) + \ell(\mu) - \ell(\nu).$$

Example

Let $\lambda = (2), \mu = (3,1), \nu = (3,1)$.



$$D^{\nu}_{\lambda,\mu} = 2^{1-2} z^2 \cdot d^{\nu}_{\lambda,\mu} = 2^{-1} z^2 \cdot 2 = z^2.$$

Conjectural rule for $L^{\gamma}_{\lambda,\mu}$

Theorem (Commutativity) [R-Yadav-Yong (2019)]

$$D^{\gamma}_{\lambda,\mu} = D^{\gamma}_{\mu,\lambda}.$$

Theorem (Localization) [R-Yadav-Yong (2019)]

$$D^{\lambda}_{\lambda,(p)} = L^{\lambda}_{\lambda,(p)} \text{ and } D^{\rho_n}_{\rho_n,\rho_n} = L^{\rho_n}_{\rho_n,\rho_n}.$$

Conjecture (Integrality) [R-Yadav-Yong (2019)]

$$D^{\gamma}_{\lambda,\mu} \in \mathbb{Z}[z].$$

Main Conjecture [R-Yadav-Yong (2019)]

$$D^{\gamma}_{\lambda,\mu} = L^{\gamma}_{\lambda,\mu}.$$

Conclusions

- Shifted tableaux compute structure constants in $H^*(Z)$.
- Edge labeled tableaux compute structure coefficients in $H^*_T(X)$.
- Shifted edge labeled tableaux conjecturally compute structure coefficients in a specialization of $H^*_T(Z)$.