Bow varieties

bow

arrows
(a.k.a. homomorphisms)

quiver

- joint work with Yiyan Shou
- learned about branes from Lev Rozansky
- related works with

  Andrey Smirnov
  Alexander Varchenko
  Zijun Zhou
  Andrzej Weber

At the beginning there were (type-A) homogeneous spaces...

$Gr_2 \mathbb{C}^4$

$\mathcal{F}_{2,5,7}$

$\mathcal{F}_{1,2,3,4}$

... and we did Schubert Calculus on them.

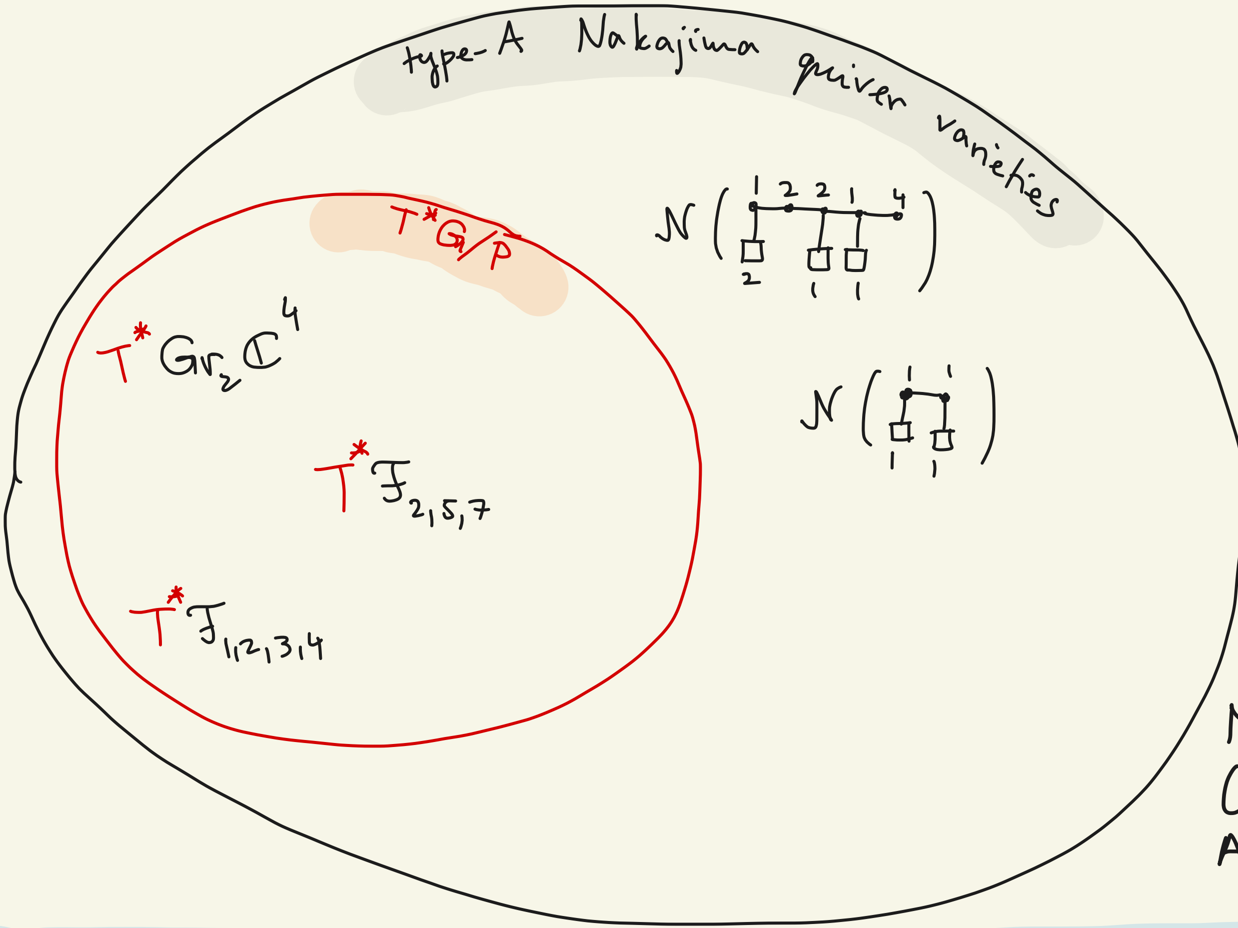
Schubert Calculus $\rightarrow$ cotangent Schubert Calculus

(a.k.a. $\hbar$- deformed Schubert Calculus)

$T^*Gr_2\mathbb{C}^4$

$T^*\mathcal{F}_{2,5,7}$

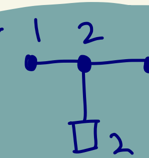
$T^*\mathcal{F}_{1,2,3,4}$



Maulik–Okounkov
Okounkov
Aganagić-Okounkov

... Schubert classes ("stable envelopes") on $\mathcal{N}(\text{quiver})$

THE COINCIDENCE !!!

$T^*\mathrm{Gr}_2\mathbb{C}^4$ $\longleftrightarrow$ $\mathcal{N}\Big($  $\Big)$

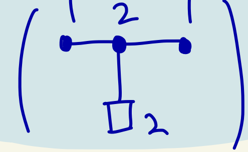
[RSVZ 2020]

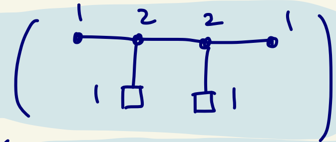
intimate relationship between their Schubert Calculus

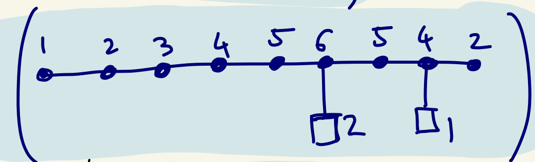
"$N=4$ $d=3$ mirror symmetry for characteristic classes"

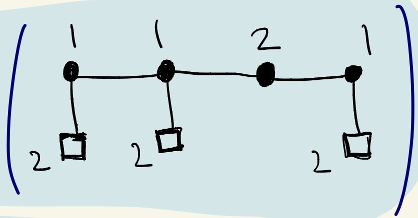
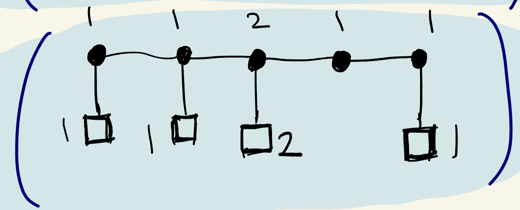
dim = 8
\# fix pts = 6
T^4 action

dim = 4
\# fix pts = 6
T^2 action

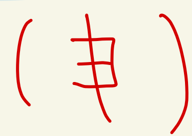
$T^*Gr_2\mathbb{C}^4 \longleftrightarrow \mathcal{N}$  [RSVZ]

$T^*Gr_2\mathbb{C}^5 \longleftrightarrow \mathcal{N}$ 

$T^*\mathcal{F}_{2,6,10} \longleftrightarrow \mathcal{N}$ 

$\mathcal{N}$  $\longleftrightarrow \mathcal{N}$ 

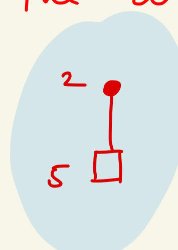
$T^*G/B \longleftrightarrow T^*G^L/B^L$ [RW 2020]

$T^*\mathcal{F}_{2,5,7} \longleftrightarrow \mathcal{N}$ 

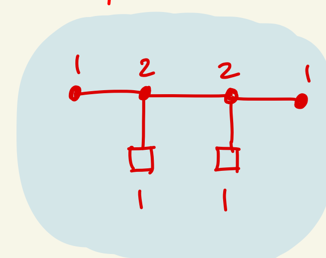
1. What exactly is the relationship between Schubert classes of 3d mirror dual spaces?

2. How to find the 3d mirror dual?

(ie what is the combinatonics that connects

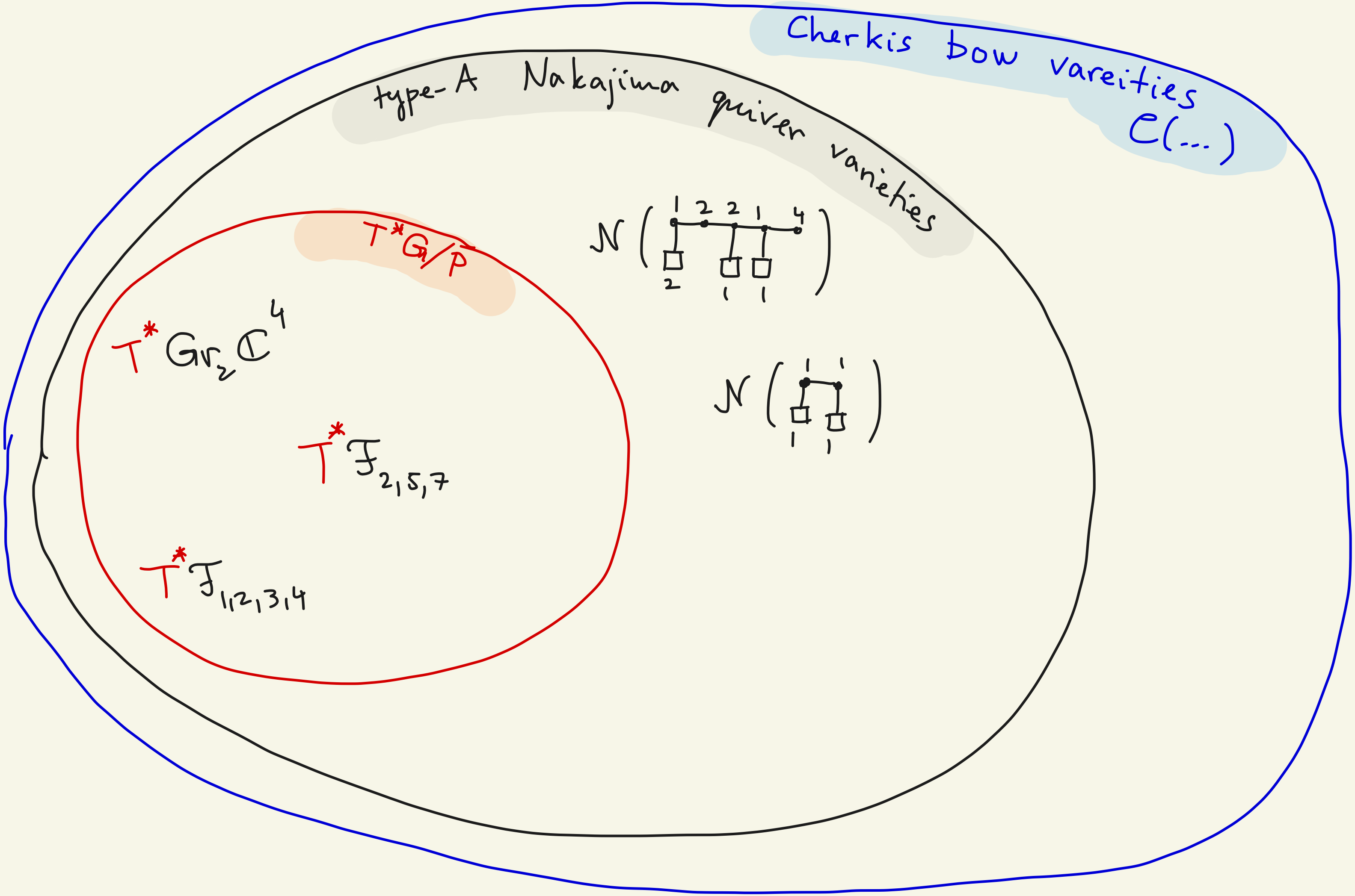


with

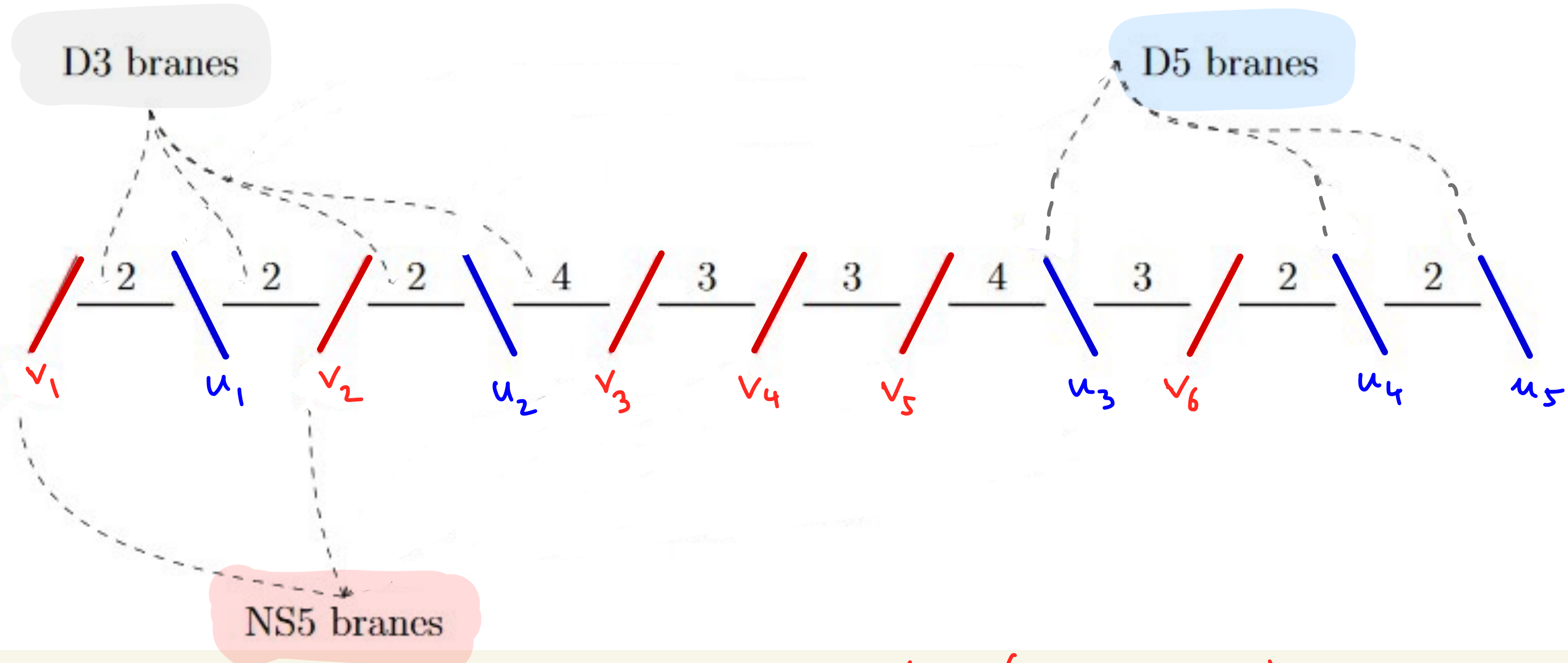


?)

(what is the mirror of $T^*\mathcal{F}_{2,5,7}$?)


Cherkis bow vareities
$\mathcal{C}(\dots)$
type-A Nakajima quiver varieties
T^*G/P
$T^*Gr_2\mathbb{C}^4$
$T^*\mathcal{F}_{2,5,7}$
$T^*\mathcal{F}_{1,2,3,4}$

Brane diagrams



v_i : Kähler (dynamical) variables

u_i : equivariant variables

brane
diagram
$\mathcal{D}$

tautological bundles,
one for each D3 brane

$C(\mathcal{D})$

$\top$ D5 branes

Cherkis bow
variety

<u>Cherkis:</u>
moduli space of
unitary instantons
on multi-Taub-NUT
spaces
(key: Nahm's
equation)

<u>Nakajima-Takayama</u>
Hamiltonian reduction
of representations
of certain quivers
with relations

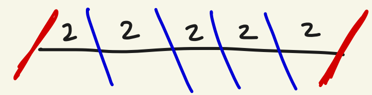
$\sim$

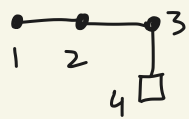
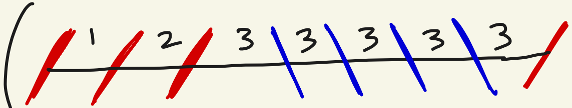
<u>Rozansky-R</u>
"symplectic
intersection"
of generalized
Lagrange
varieties

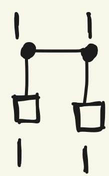
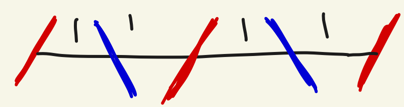
How are $\mathcal{N}$(quiver) special cases?

<u>Examples</u>

$T^*\mathbb{P}^1 = \mathcal{N}\Big($  $\Big) = \mathcal{C}\Big($  $\Big)$

$T^*Gr_2\mathbb{C}^4 = \mathcal{N}\Big($  $\Big) = \mathcal{C}\Big($  $\Big)$

$T^*\mathcal{F}_{1,2,3,4} = \mathcal{N}\Big($  $\Big) = \mathcal{C}\Big($  $\Big)$

$\mathcal{N}\Big($  $\Big) = \mathcal{C}\Big($  $\Big)$

<u>Observe</u> 

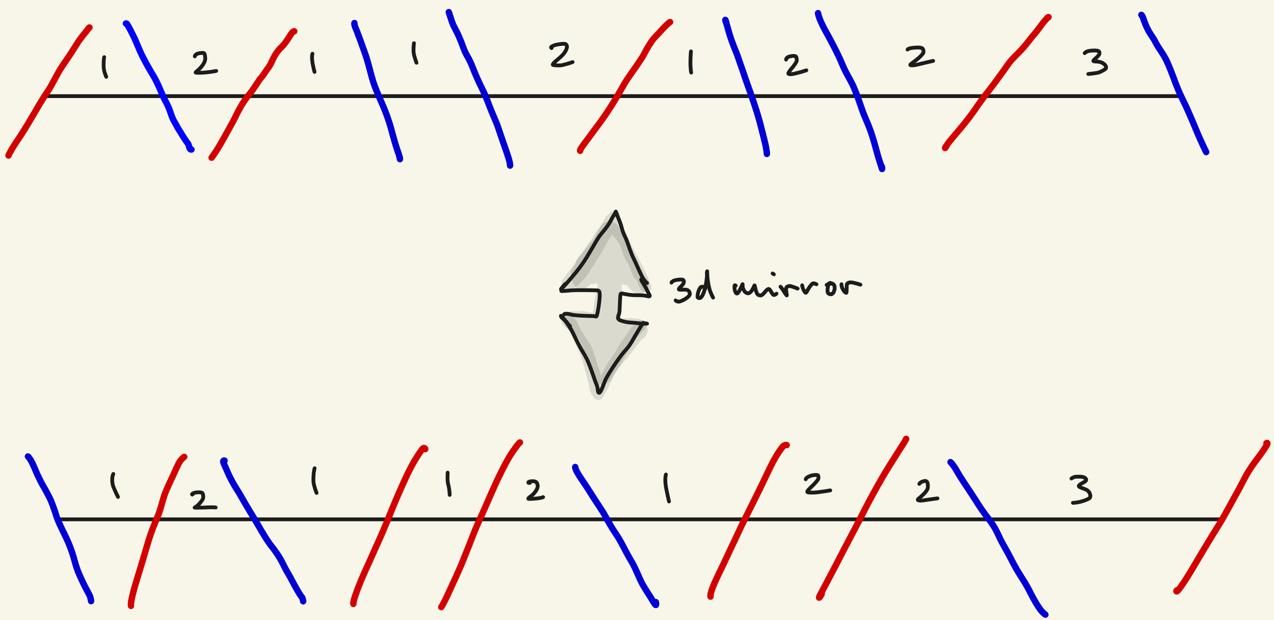
"cobalanced brane diagram"

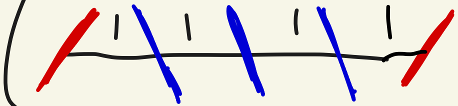
There is a <u>lot</u> to say about $C(D)$ spaces.

Today:

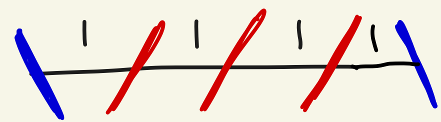
- 3D mirror symmetry
- Hanany-Witten (HW) transition
- combinatorics of torus fixed points

3D mirror symmetry for bow varieties:



Ex $\quad T^*\mathbb{P}^2 = \mathcal{N}\left(\begin{array}{c} \overset{1}{\bullet} \\ | \\ \square_3 \end{array}\right) = \mathcal{C}\Big($  $\Big)$ dim 4

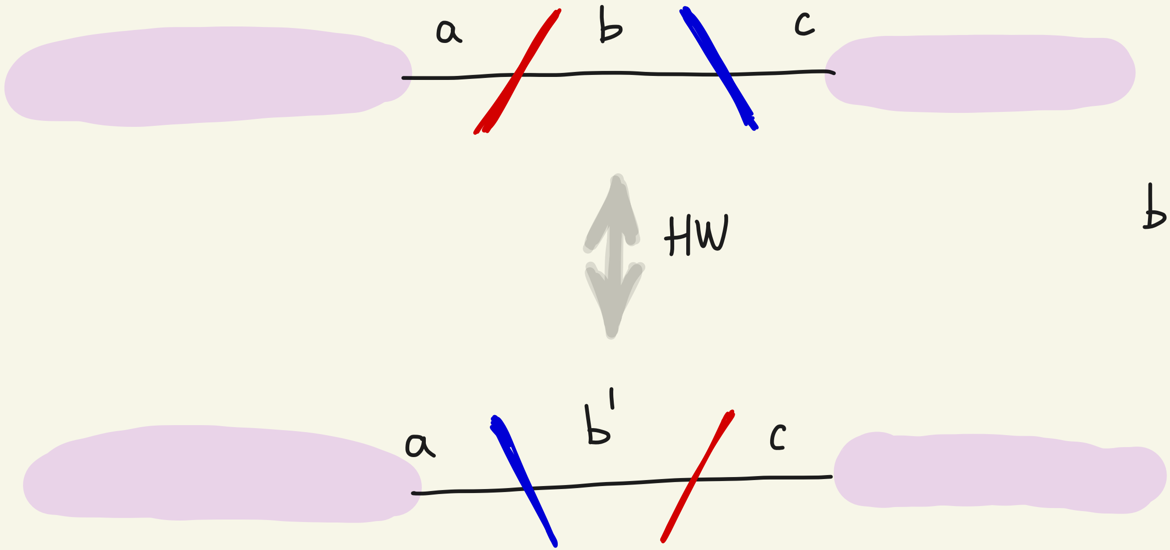
$\Updownarrow$ 3d mirror

$\mathcal{C}\Big($  $\Big)$ dim 2

not cobalanced, ie not $\mathcal{N}(\dots)$

... but ... <to be continued>

Hanany-Witten transition on brane diagrams.

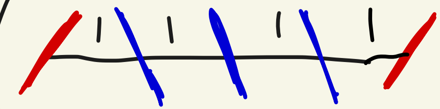


$$b + b' = a + c + 1$$

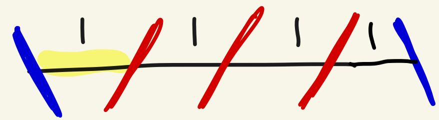
(why? later: "brane charge")

<u>Thm</u> $\mathcal{C}(\mathcal{D}) \cong \mathcal{C}(HW(\mathcal{D}))$

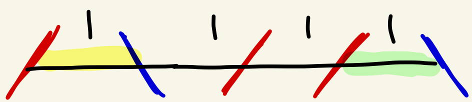
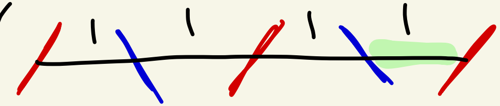
Ex

$T^*\mathbb{P}^2 = \mathcal{N}\Big($  $\Big) = \mathcal{C}\Big($  $\Big)$

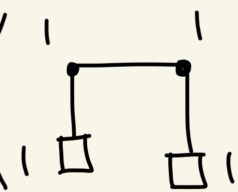
$\Updownarrow$ 3d mirror

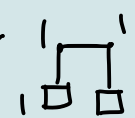
$\mathcal{C}\Big($  $\Big)$

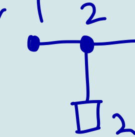
HW

$\mathcal{C}\Big($  $\Big) \overset{HW}{=} \mathcal{C}\Big($  $\Big)$

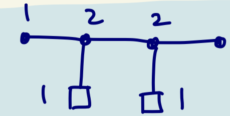
$\|$

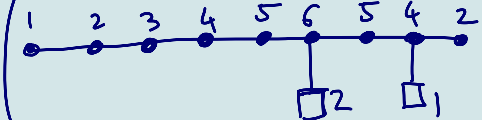
$\mathcal{N}\Big($  $\Big)$

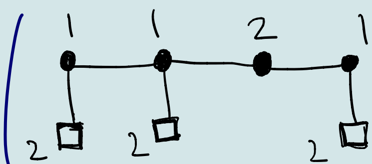
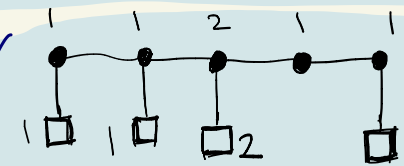
$\Rightarrow \quad T^*\mathbb{P}^2 \xleftrightarrow{\text{3d mirror}} \mathcal{N}\Big($  $\Big)$

$T^*Gr_2\mathbb{C}^4 \longleftrightarrow \mathcal{N}\Big($  $\Big)$

[RSVZ]

$T^*Gr_2\mathbb{C}^5 \longleftrightarrow \mathcal{N}\Big($  $\Big)$

$T^*\mathcal{F}_{2,6,10} \longleftrightarrow \mathcal{N}\Big($  $\Big)$

$\mathcal{N}\Big($  $\Big) \longleftrightarrow \mathcal{N}\Big($  $\Big)$

$T^*G/B \longleftrightarrow T^*G^L/B^L$

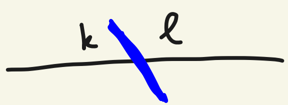
[RW 2020]

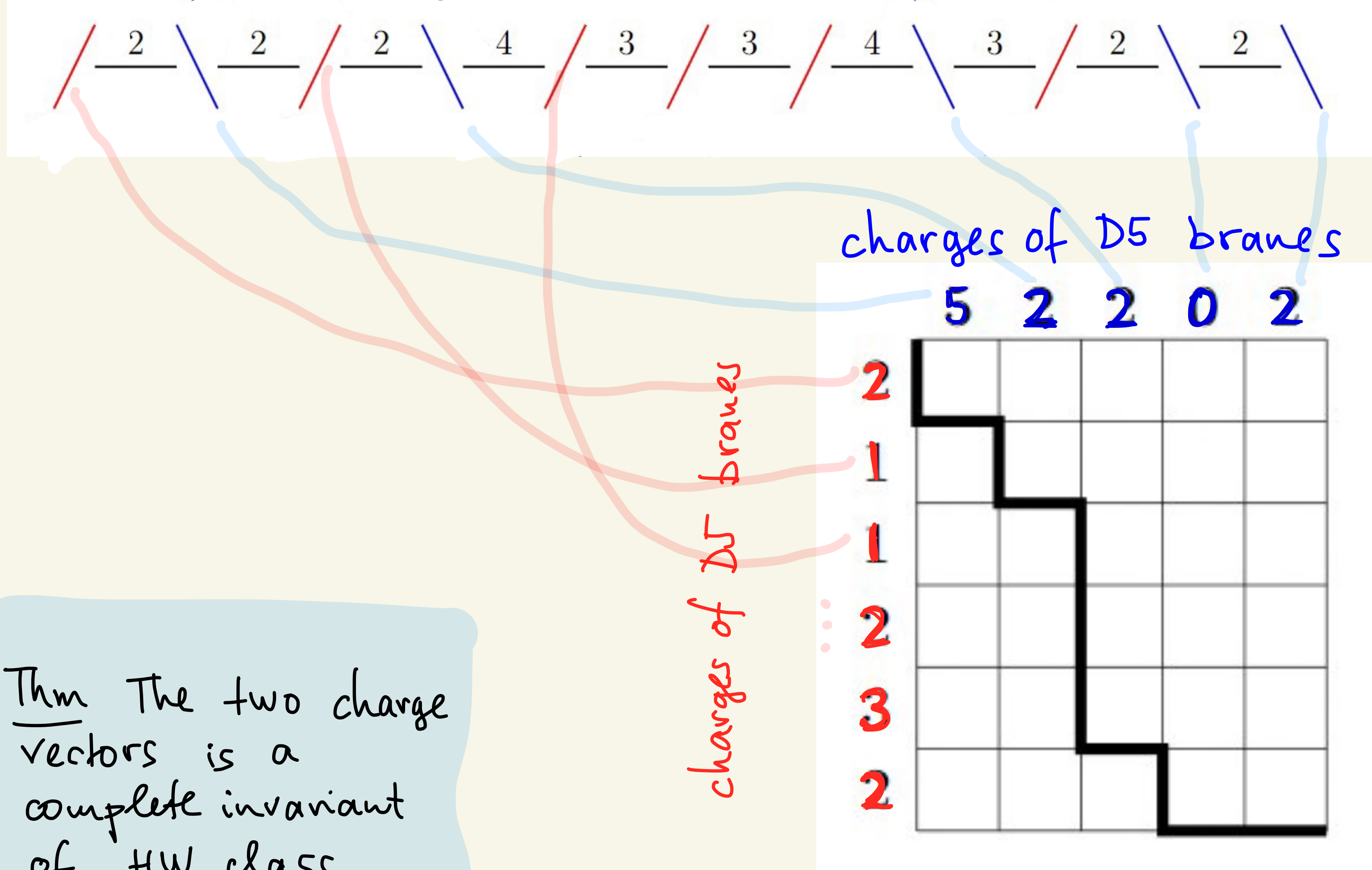
$T^*\mathcal{F}_{2,5,7} \longleftrightarrow \mathcal{N}\Big($  $\Big)$

def brane charge

$$\text{charge}\left(\text{NS5 brane}\right) := \ell - k + \#\{\text{D5-branes left of it}\}$$

$$\text{charge}\left(\text{D5 brane}\right) := k - \ell + \#\{\text{NS5-branes right of it}\}$$



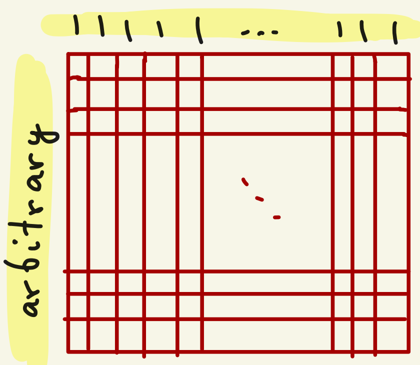


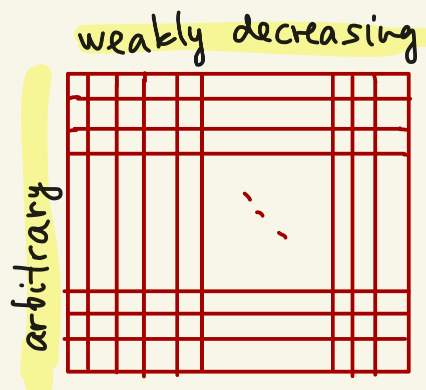
<u>Thm</u> The two charge vectors is a complete invariant of HW class

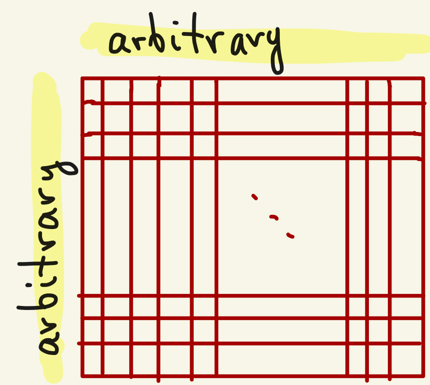
Thm (up to HW transitions)

$$T^*G/P \subset \mathcal{N}(\text{quiver}) \subset \mathcal{C}(\text{brane diagram})$$

$\updownarrow$ $\updownarrow$ $\updownarrow$



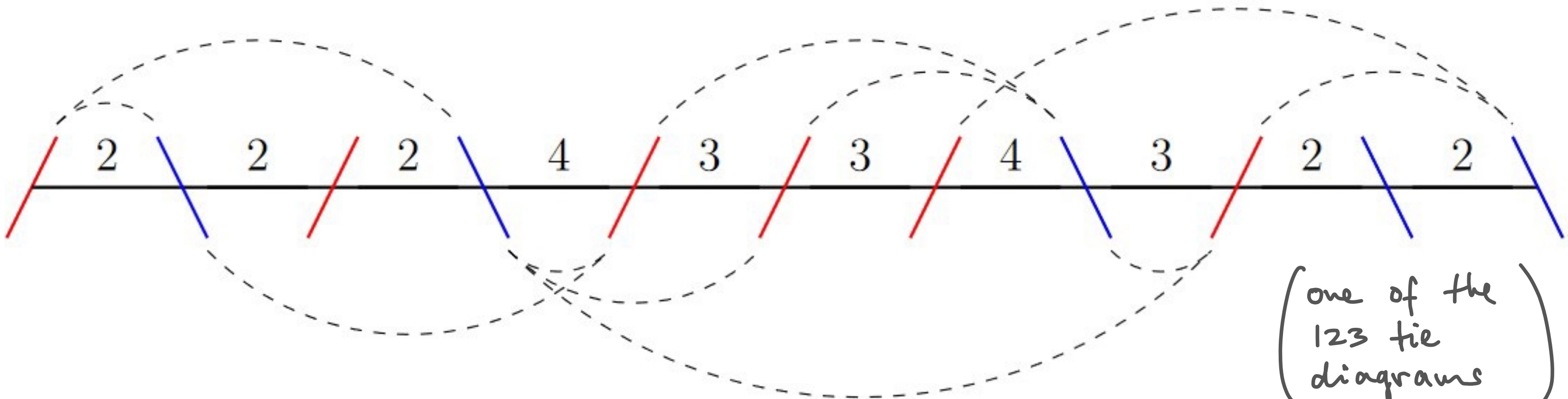




Torus fixed points on bow varieties

- tie diagrams
- binary contingency tables

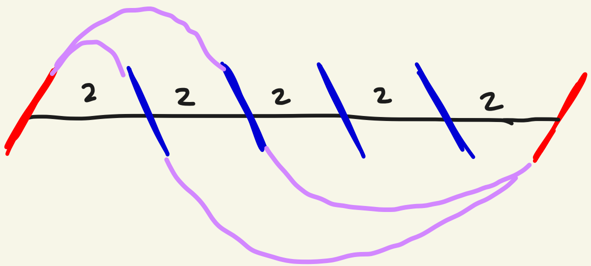
[redacted] AirSpeed EXACT Reach AS3008A Upright Vacuum, Bagless, Allergy Filter, Blue/Black fixed points $\xleftrightarrow{1:1}$ tie diagrams

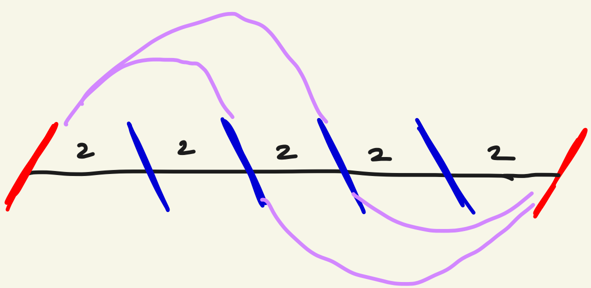


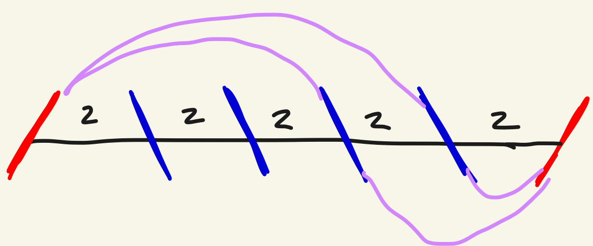
(one of the 123 tie diagrams)

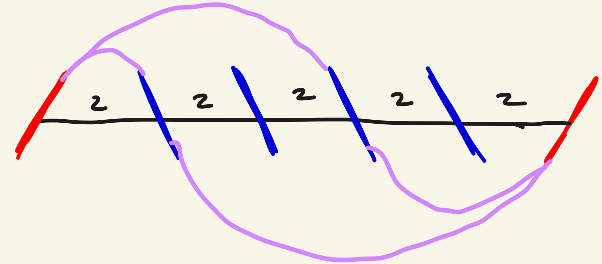
- a tie must connect 5-branes of different kinds
- each D3 brane to be coved as many times as its multiplicity

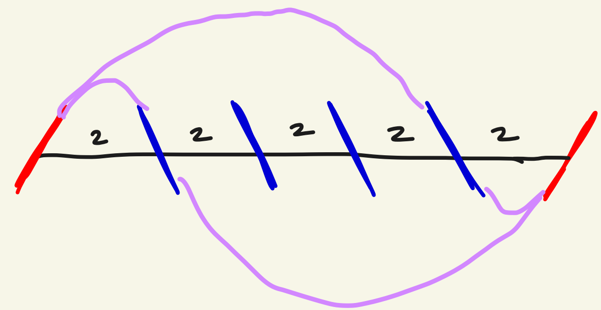
fixed points of $T^*Gr_2\mathbb{C}^4$:

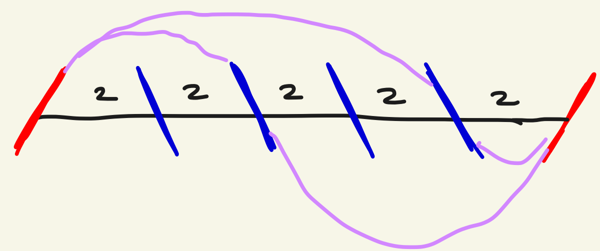


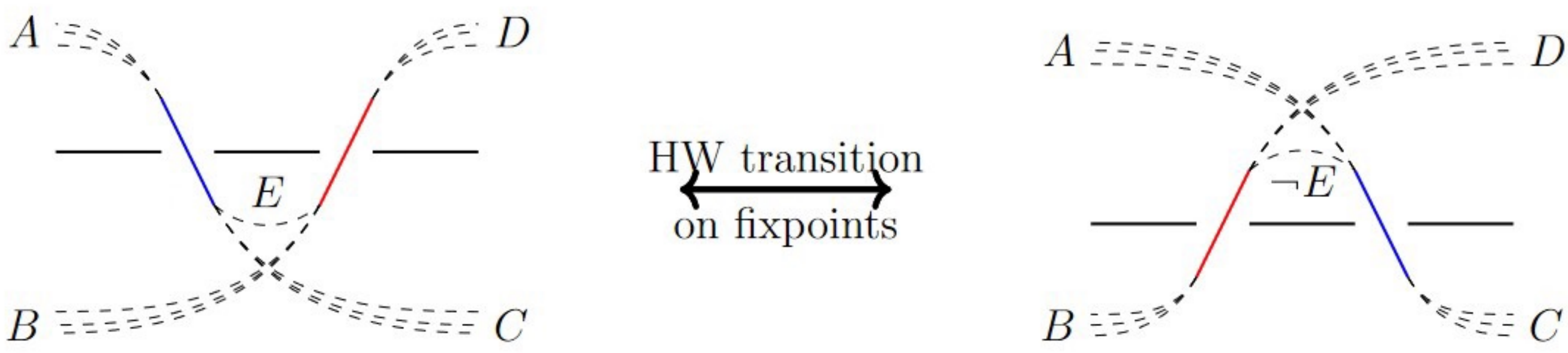




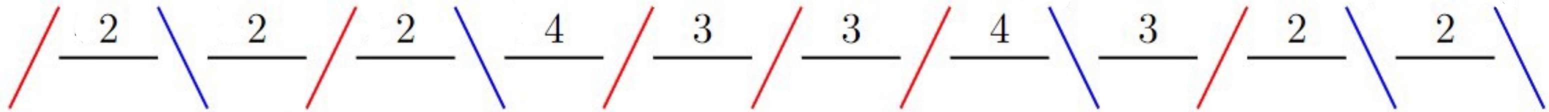








$\sim$ Reidemeister-III



BCT : 0-1-matrix with row & column sums the charge vectors

Thm

fix pts $\overset{1:1}{\longleftrightarrow}$ BCT's

one of the 123 BCTs

$$
\begin{array}{c|ccccc}
 & 5 & 2 & 2 & 0 & 2 \\
\hline
2 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 \\
2 & 1 & 0 & 1 & 0 & 0 \\
3 & 1 & 1 & 0 & 0 & 1 \\
2 & 1 & 0 & 0 & 0 & 1
\end{array}
$$

fix pts of $T^*\mathrm{Gr}_2\mathbb{C}^4$ as BCT's

$$\begin{array}{c|c|c|c|c|} & 1 & 1 & 1 & 1 \\ \hline 2 & & & & \\ \hline 2 & & & & \\ \hline \end{array}$$

$$\begin{array}{|c|} 1 \\ \hline \\ \hline \\ \hline \end{array} \text{ must be } \begin{array}{|c|} 1 \\ \hline 1 \\ \hline 0 \\ \hline \end{array} \text{ or } \begin{array}{|c|} 1 \\ \hline 0 \\ \hline 1 \\ \hline \end{array}$$

$$2 \begin{array}{|c|} \hline 1 \\ \hline 0 \\ \hline \end{array} \text{ and } 2 \begin{array}{|c|} \hline 0 \\ \hline 1 \\ \hline \end{array}$$

$\rightarrow \binom{4}{2}$ fixed points

So far:

- $\mathcal{C}($ brane diagram $)$

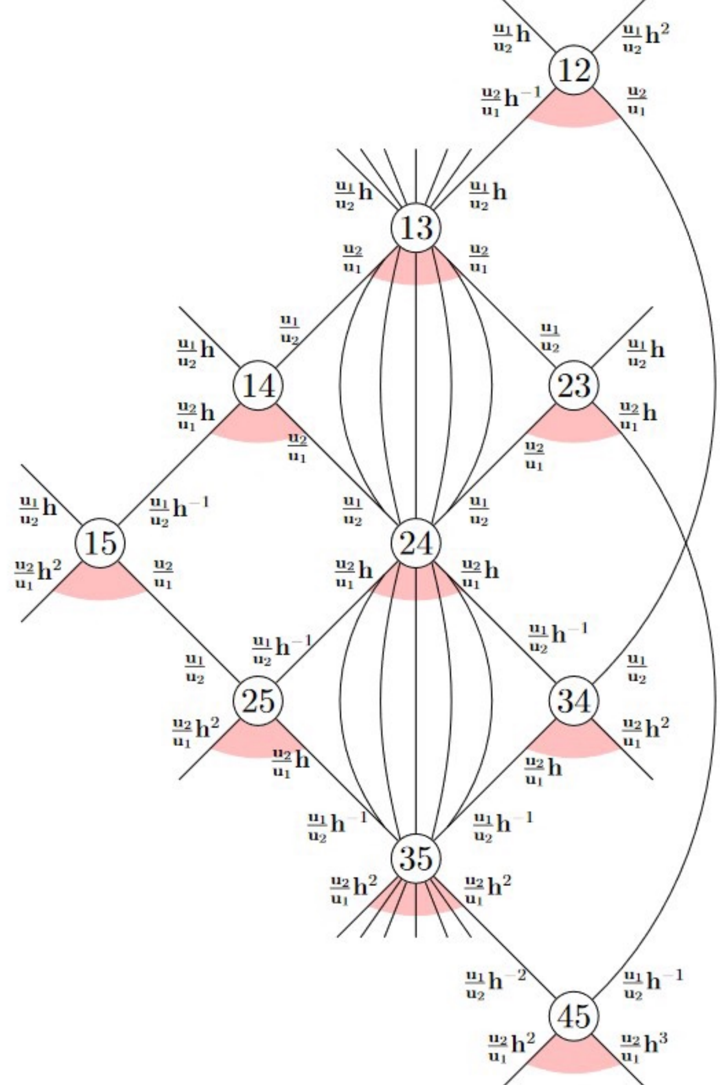
  operations : - 3d mirror

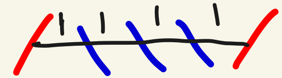
  - HW transition

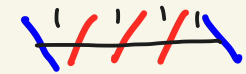
  brane charge

  combinatorics of torus fixed pts.

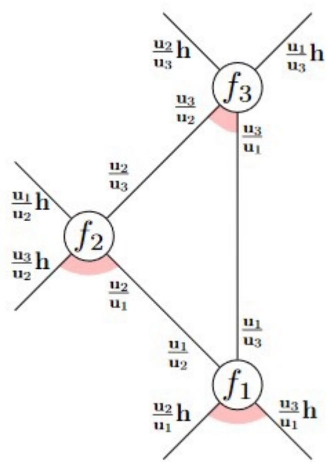
3d mirror of $T^*Gr_2\mathbb{C}^5$

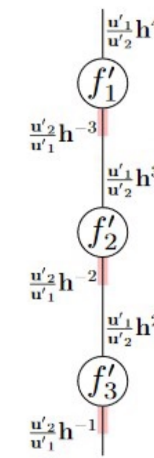




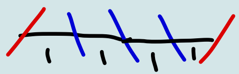


$T^*\mathbb{P}^2$ $\longleftrightarrow$ its mirror

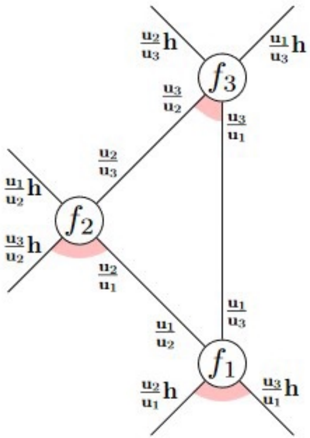




$T^*\mathbb{P}^2 = \mathcal{N}\Big($  $\Big)$

$= \mathcal{C}\Big($  $\Big)$

	f_1	f_2	f_3
f_1	$\theta(\frac{u_1}{u_2})\theta(\frac{u_1}{u_3})\theta(\frac{v_2}{v_1}h^4)$	0	0
f_2	$\theta(h)\theta(\frac{u_1}{u_3})\theta(\frac{u_2 v_2}{u_1 v_1}h^3)$	$\theta(\frac{u_1}{u_2}h)\theta(\frac{u_2}{u_3})\theta(\frac{v_2}{v_1}h^3)$	0
f_3	$\theta(h)\theta(\frac{u_2}{u_1}h)\theta(\frac{u_3 v_2}{u_1 v_1}h^2)$	$\theta(h)\theta(\frac{u_1}{u_2}h)\theta(\frac{u_3 v_2}{u_2 v_1}h^2)$	$\theta(\frac{u_2}{u_3}h)\theta(\frac{u_1}{u_3}h)\theta(\frac{v_2}{v_1}h^2)$

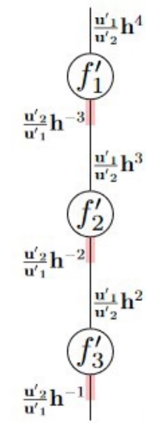


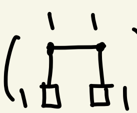
<u>Thm</u>: same after

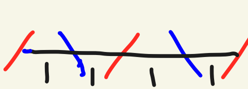
$(\ )^T$

$u_\cdot \longleftrightarrow v_\cdot$

$h \longleftrightarrow h^{-1}$



$\mathcal{N}\Big($  $\Big) =$

$\mathcal{C}\Big($  $\Big)$

	f'_1	f'_2	f'_3
f'_1	$\theta(\frac{u'_1}{u'_2}h^4)\theta(\frac{v'_2}{v'_1})\theta(\frac{v'_3}{v'_1})$	$\theta(h)\theta(\frac{v'_3}{v'_1})\theta(\frac{v'_2 u'_2}{v'_1 u'_1}h^{-3})$	$\theta(h)\theta(\frac{v'_2}{v'_1}h^{-1})\theta(\frac{v'_3 u'_2}{v'_1 u'_1}h^{-2})$
f'_2	0	$\theta(\frac{u'_1}{u'_2}h^3)\theta(\frac{v'_2}{v'_1}h)\theta(\frac{v'_3}{v'_2})$	$\theta(h)\theta(\frac{v'_2}{v'_1}h)\theta(\frac{v'_3 u'_2}{v'_2 u'_1}h^{-2})$
f'_3	0	0	$\theta(\frac{u'_1}{u'_2}h^2)\theta(\frac{v'_3}{v'_2}h)\theta(\frac{v'_3}{v'_1}h)$