Doppelgänger posets and the K-theory of flag varieties

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University of Waterloo

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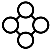
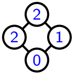
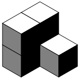
Special Session on
Recent Advances in Schubert Calculus and Related Topics
March 2021

Based on joint work with
Zachary Hamaker (Florida), Rebecca Patrias (St. Thomas), and Nathan Williams (UT Dallas)

Plane partitions

- Consider the poset $\mathcal{P} =$ 
- A **plane partition** (of height ℓ) over $\mathcal{P}$ is a weakly order-preserving map $\mathcal{P} \to \{0, 1, \ldots, \ell\}$





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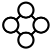
- Consider the poset $\mathcal{P} =$ 
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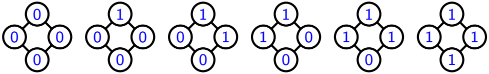
$\mathrm{PP}^{[1]}(\mathcal{P}) =$ 

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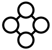
- Consider the poset $\mathcal{P} =$ 
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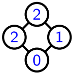

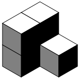

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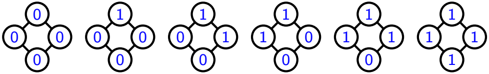
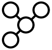
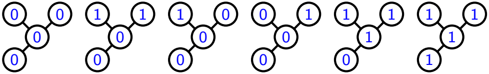
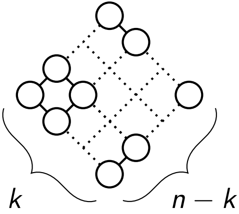
$\mathrm{PP}^{[1]}(\mathcal{P}) =$ 

- **Ex:** Plane partitions of height 1 over $\mathcal{Q} =$ :

$\mathrm{PP}^{[1]}(\mathcal{Q}) =$ 

Doppelgängers

- Let $\Lambda_{Gr(k,n)} =$ 

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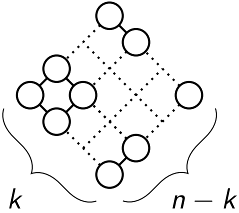
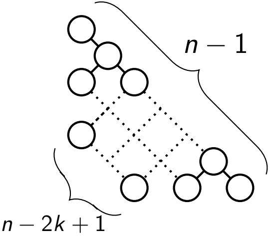
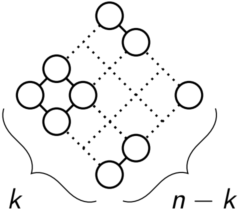
- Let $\Lambda_{\mathrm{Gr}(k,n)} =$  **(rectangle)**

- $\Phi^{+}_{B_{k,n}} =$  **(trapezoid)**

- Let $\Lambda_{\mathrm{Gr}(k,n)} =$ 

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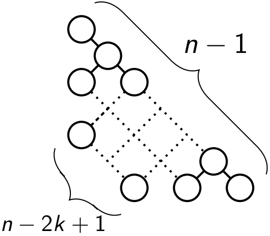
- $\Phi^+_{B_{k,n}} =$ 

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Theorem (Proctor, 1983)

For all ℓ, $\mathrm{PP}^{[\ell]}\left(\Lambda_{\mathrm{Gr}(k,n)}\right) \cong \mathrm{PP}^{[\ell]}\left(\Phi^+_{B_{k,n}}\right)$

Combinatorial proof?

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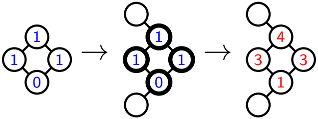
Theorem (Hamaker+Patrias+P+Williams 2020)

For all ℓ, explicit bijections $\mathsf{PP}^{[\ell]}\left(\Lambda_{\mathrm{Gr}(k,n)}\right) \cong \mathsf{PP}^{[\ell]}\left(\Phi^+_{B_{k,n}}\right)$ are given via the combinatorics of K-theoretic Schubert calculus.

Example bijection

Convert to increasing tableau:



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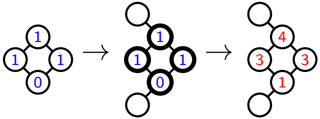
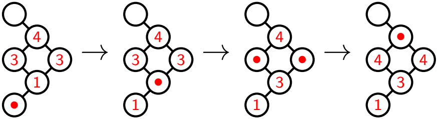


K-**jeu de taquin:**



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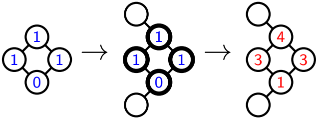

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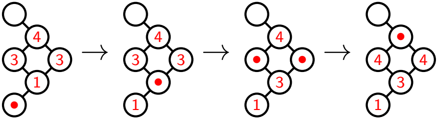
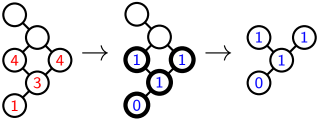
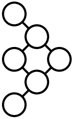


Convert back to PP:



The secret geometry

- The ambient poset  is $\Lambda_{\mathrm{OG}(n,2n)}$, which describes the Schubert decomposition of the **orthogonal Grassmannian** $\mathrm{OG}(n, 2n)$ parametrizing isotropic n-planes in $\mathbb{C}^{2n}$.

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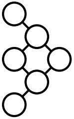
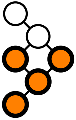
- The ambient poset  is $\Lambda_{\mathrm{OG}(n,2n)}$, which describes the Schubert decomposition of the **orthogonal Grassmannian** $\mathrm{OG}(n, 2n)$ parametrizing isotropic n-planes in $\mathbb{C}^{2n}$.
- The embedded trapezoid  indexes a particular Schubert variety $X_w \hookrightarrow \mathrm{OG}(n, 2n)$

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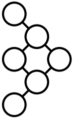
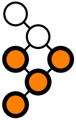
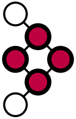
- The ambient poset  is $\Lambda_{OG(n,2n)}$, which describes the Schubert decomposition of the **orthogonal Grassmannian** $OG(n, 2n)$ parametrizing isotropic n-planes in $\mathbb{C}^{2n}$.
- The embedded trapezoid  indexes a particular Schubert variety $X_w \hookrightarrow OG(n, 2n)$
- The embedded rectangle  indexes a particular Richardson variety $X_u^v = X_u \cap X^v \hookrightarrow OG(n, 2n)$

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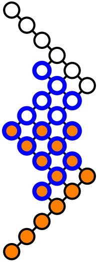
- Generalizes to other spaces. . .

. . . such as

- Let $\Lambda_{\mathrm{OG}(6,12)}$ be the thick blue-circled nodes of 

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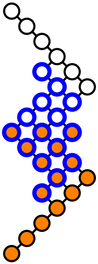

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Corollary (Hamaker+Patrias+P+Williams 2020)

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- Comes from analogous geometry on the E_7 minuscule variety

Thanks!

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