Quasi-split symmetric pairs of $\mathfrak{sl}_n$ and Steinberg varieties of classical type

Yiqiang Li

State University of New York at Buffalo

AMS special session on modern Schubert Calculus

Quasi-split symmetric pairs of $sl_n(\mathbb{C})$ & Steinberg varieties of classical type

(I) Quasi-split symmetric pairs

Let $\mathfrak{g}$ be a complex s.s. Lie alg.

Let $\theta : \mathfrak{g} \to \mathfrak{g}$ be an involution.

Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, $\mathfrak{k} = \{x \in \mathfrak{g} \mid \theta(x) = x\}$, $\mathfrak{p} = \{x \in \mathfrak{g} \mid \theta(x) = -x\}$.

$(\mathfrak{g}, \mathfrak{k})$ is called a symmetric pair.

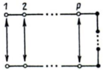
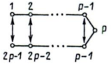
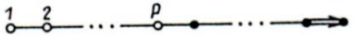
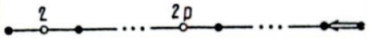
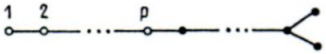
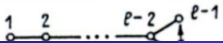
(i) Classification of $(\mathfrak{g}, \mathfrak{k})$ $\cong$ Classification of real simple Lie algebras, due to Élie Cartan.

They are parametrized by Satake diagrams, i.e., bicolored Dynkin diagrams with graph involutions.

Quasi-split are those whose Satake diagrams are in white colors.

ONISHCHIK -VINBERG :

Table 4. ~~The table~~ lists noncompact real Lie algebras $\mathfrak{g}$ that do not admit a complex structure, i.e. the real form of complex simple Lie algebras $\mathfrak{g}(\mathbb{C})$. The table also indicates the type of the system Σ of real roots and the restriction map $r\colon \Pi_1 \to \Theta$, where Θ is the system of simple roots of Σ. The simple roots from Π are denoted by α_j, those from Θ by λ_j; the numbering in both these systems is the same as in Table 1.

$\mathfrak{g}(\mathbb{C})$	$\mathfrak{g}$	$\mathfrak{k}$	$\dim\mathfrak{k}$	$\dim\mathfrak{p}$	$\mathrm{rk}_{\mathbb{R}}\mathfrak{g}$	Satake diagram
$\mathfrak{sl}_{l+1}(\mathbb{C})$ $(l \geqslant 1)$	$\mathfrak{sl}_{l+1}(\mathbb{R})$	$\mathfrak{so}_{l+1}$	$\frac{1}{2}l(l+1)$	$\frac{1}{2}l(l+3)$	l	
	$\mathfrak{sl}_{l+1}(\mathbb{H})$ $(l=2p+1, p \geqslant 1)$	$\mathfrak{sp}_{p+1}$	$(p+1)\times(2p+3)$	$p(2p+3)$	p	
	$\mathfrak{su}_{p,l+1-p}$ $(1 \le p \le \frac{l}{2})$	$\mathfrak{su}_p \oplus \mathfrak{u}_{l+1-p}$	$p^2+(l+1-p)^2-1$	$2p\times(l+1-p)$	p	
	$\mathfrak{su}_{p,p}$ $(l=2p-1, p \geqslant 2)$	$\mathfrak{su}_p \oplus \mathfrak{u}_p$	$2p^2-1$	$2p^2$	p	
$\mathfrak{so}_{2l+1}(\mathbb{C})$ $(l \geqslant 1)$	$\mathfrak{so}_{p,2l+1-p}$ $(1 \le p \le l)$	$\mathfrak{so}_p \oplus \mathfrak{so}_{2l+1-p}$	$p(2p+1)-(2l+1-p)(4l+3-2p)$	$p(2l+1-p)$	p	
$\mathfrak{sp}_{2l}(\mathbb{C})$	$\mathfrak{sp}_{2l}(\mathbb{R})$	$\mathfrak{u}_l$	l^2	$l(l+1)$	l	
	$\mathfrak{sp}_{p,l-p}$ $(1 \le p \le \frac{1}{2}(l-1))$	$\mathfrak{sp}_p \oplus \mathfrak{sp}_{l-p}$	$p(2p+1)+(l-p)\times(2l-2p+1)$	$4p(l-p)$	p	
	$\mathfrak{sp}_{p,p}$ $(l=2p)$	$\mathfrak{sp}_p \oplus \mathfrak{sp}_p$	$2p(2p+1)$	$4p^2$	p	
$\mathfrak{so}_{2l}(\mathbb{C})$ $(l \geqslant 4)$	$\mathfrak{so}_{p,2l-p}$ $(1 \le p \le l-2)$	$\mathfrak{so}_p \times \mathfrak{so}_{2l-p}$	$\frac{1}{2}p(p-1)+\frac{1}{2}(2l-p)\times(2l-p+1)$	$p(2l-p)$	p	
	$\mathfrak{so}_{l-1,l+1}$	$\mathfrak{so}_{l-1}\times\mathfrak{so}_{l+1}$	$\frac{1}{2}(l-1)\times(l-2)+$	l^2-1	$l-1$	

Let $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$, $\theta : e_i \mapsto f_{n-i}$,
$f_i \mapsto e_{n-i}$
$h_i \mapsto -h_{n-i}$,

Then $(\mathfrak{sl}_n(\mathbb{C}), \mathfrak{sl}_n(\mathbb{C})^\theta)$ is a symmetric pair of quasi-split type Aiii.

For ADE TYPE : Quasi-split : Ai, Aiii/iv, DI, EI, EII, EV, EVIII.

(ii) Conjecture (Li). Certain twisted quiver varieties provide a geometric setting for the quasi-split symmetric pairs of ADE type.

Today I'll talk about a proof of this conjecture in the case $(\mathfrak{sl}_n(\mathbb{C}), \mathfrak{sl}_n(\mathbb{C})^\theta)$.

(II) Steinberg varieties of classical type.

Let W be a complex vector space of dimension w.

Fix $\varepsilon \in \{\pm 1\}$. Let $\langle -,- \rangle : W \times W \to \mathbb{C}$ be a non-deg bilinear ε-form, i.e., $\langle w_1, w_2 \rangle = \varepsilon \langle w_2, w_1 \rangle$, $\forall\, w_1, w_2 \in W$.

Let $G \equiv G_{W,\varepsilon}$ = the isometry group w.r.t. $\langle -,- \rangle$.

Let $\mathcal{F}_{n,w,\varepsilon} = \{ n\text{-step isotropic flags in } W \}$

$$= \{ 0 \equiv F_0 \subseteq F_1 \subseteq \cdots \subseteq F_n \equiv W : F_i^{\perp} = F_{n-i} \ \forall i \in [0,n] \}.$$

$$G \curvearrowright \mathcal{F}_{n,w,\varepsilon} = \coprod_{\substack{\lambda = (\lambda_i)_{1 \le i \le n} \\ \lambda_1 + \cdots + \lambda_n = w \\ \lambda_i = \lambda_{n-i}}} \mathcal{F}_\lambda, \quad \mathcal{F}_\lambda = \{ F \in \mathcal{F}_{n,w,\varepsilon} : \dim F_i/F_{i-1} = \lambda_i \}$$

$$= \coprod_{\lambda} G/P_\lambda, \quad P_\lambda = \mathrm{stab}_G(F), \ F \in \mathcal{F}_\lambda.$$

The G action induces a moment map

$$\mu : T^* \mathcal{F}_{n,w,\varepsilon} \longrightarrow (\mathrm{Lie}\, G)^*$$

Let $Z_{n,w,\varepsilon} = T^* \mathcal{F}_{n,w,\varepsilon} \times_{(\mathrm{Lie}\, G)^*} T^* \mathcal{F}_{n,w,\varepsilon}$ ... n-step Steinberg varieties of classical type.

THM (Li) $\exists$ alg. homo $U(sl_n^\theta) \longrightarrow H^{BM}_{top}(Z_{n,w,\varepsilon})$.

The image of the chevalley generators of sl_n^θ can be described explicitly. First of all, $sl_n^\theta = \langle e_{i,\theta}, h_{i,\theta} \rangle$,

where $e_{i,\theta} = e_i + f_{n-i} \quad 1 \le i \le n-1$,

$h_{i,\theta} = h_i - h_{n-i} \quad 1 \le i \le n-1$.

For n odd, $Z_i = \left\{ (x, F, F') \in Z_{\eta,w,\varepsilon} : \begin{array}{l} F_j = F_j' \quad j \ne i, n-i \\ F_i \supseteq F_i', \ \dim F_i/F_i' = 1 \end{array} \right\}$

A closed subvariety in $Z_{\eta,w,\varepsilon}$

$e_{i,\theta} \xmapsto[\text{a sign}]{\text{up to}} [Z_i]$ ... the fundamental class of Z_i.

For n even, Z_i is Not Well-defined for $i = n/2$.

$Z_{n/2} = \left\{ (x, F, F') \mid \begin{array}{l} F_j = F_j' \quad j \ne n/2 \\ |F_i \cap F_i'| \ge |F_i| - 1 \end{array} \right\}$

$e_{n/2,\theta} \xmapsto{\text{up to}} [Z_{n/2}]$ + diagonal

THM2. Let w varies. $\left(H^{BM}_{top}(Z_{n,w,\varepsilon})\right)_{w\in\mathbb{N}}$ forms a projective system. Then

(1) $$\mathbf{U}(\mathfrak{sl}_n^{\theta}) \hookrightarrow \varprojlim_{w} H^{BM}_{top}(Z_{n,w,\varepsilon}).$$

(2) $$\dot{\mathbf{U}}(\mathfrak{sl}_n^{\theta}) \hookrightarrow \varprojlim_{w} H^{BM}_{top}(Z_{n,w,\varepsilon})$$

(3) $\dot{\mathbf{U}}(\mathfrak{sl}_n^{\theta})$ admits a basis of topological origin.

(III) Representation theory of $U(sl_n^\theta)$

Identify $(\mathrm{Lie}\,G)^*$ with $\mathrm{Lie}\,G$. Given a nilpotent elt $x \in \mathrm{Lie}\,G$, we consider the Springer fiber

$$\mu^{-1}(x)$$

Then $sl_n^\theta \curvearrowright H_*^{BM}(\mu^{-1}(x))$. On the other hand, we can consider the pushforward $\mu_!(IC(T^*\mathcal{F}_{n,w,\varepsilon})) =: L_{n,w,\varepsilon}$

then

$$L_{n,w,\varepsilon} \cong \bigoplus_{\mu,\psi} IC(\overline{\mathcal{O}}_\mu, \mathcal{L}_\psi) \otimes W_{\mu,\psi}$$

μ a partition, ψ irred. rep. of the component group of the G-orbit $\mathcal{O}_\mu$.

THM The list of $W_{\mu,\psi}$ exhausts all rational simple modules of sl_n^θ.

"rational": all modules coming from $(\mathbb{C}^n)^{\otimes d}$, as sl_n^θ-mod.

(IV) Now we consider G an arbitrary reductive group, connected, simply connected

Z_G be a Steinberg variety defined similar to $Z_{n,\omega,\varepsilon}$.

$$K_{G\times\mathbb{C}^*}(Z_G) \quad \text{... the } \overset{G\times\mathbb{C}^*\text{-}}{\text{equivariant}} \text{ K-theory of } G.$$

Let $G^\vee$ be the Langlands dual of G. There exists an affine partial flag variety $\mathcal{P}$ corresponding to Z_G.

In light of the Langlands reciprocity for affine Hecke algebra

Grothendieck group of the bonded derived categ of $G^\vee$-eq. constructible sheaves on $\mathcal{P}\times\mathcal{P}$

<u>Question</u>: $K_{G\times\mathbb{C}^*}(Z_G) \xrightarrow{\cong} K_0(D^b_{G^\vee}(\mathcal{P}\times\mathcal{P}))$

<u>THM</u>. This holds if the underlying ring $\mathbb{Z}[q,q^{-1}]$ is extended to $\mathbb{Q}(q)$.

Thank you!