Positroid Varieties and q,t-Catalan numbers

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based on joint work with Pavel Galashin

Positroid stratification

$$Gr(k,n) = \bigsqcup_{\lambda} \Omega_\lambda$$ Schubert stratification

$$\chi : \mathbb{C}^n \longrightarrow \mathbb{C}^n \qquad e_i \longmapsto e_{\substack{i+1 \\ \text{mod } n}}$$ cyclic rotation

$$\mathring{\Pi}_f := \Omega_{\lambda(1)} \cap \chi(\Omega_{\lambda(2)}) \cap \dots \cap \chi^{n-1}(\Omega_{\lambda(n)})$$ open positroid variety

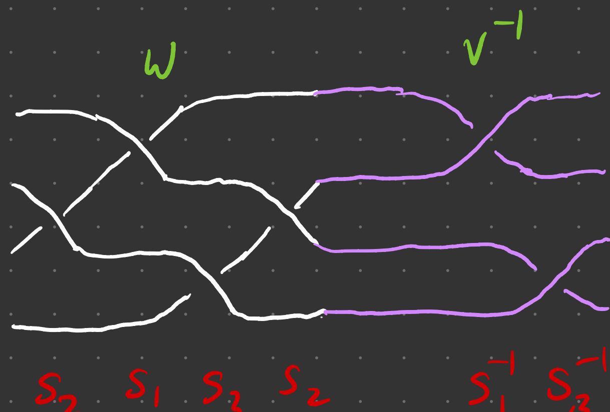
Theorem [Knutson - L. - Speyer]

- $Gr(k,n) = \bigsqcup_{f} \mathring{\Pi}_f$ positroid stratification
- $\mathring{\Pi}_f$ is smooth, irreducible, affine.
- $\mathring{\Pi}_f = \pi(\Omega_w \cap \Omega^v)$ for some $v \leq w$, $\quad Fl(n) \xrightarrow{\pi} Gr(k,n)$
- $\mathring{\Pi}_{id_{k,n}} = Gr(k,n) - \{$n rotations of Schub. divisor$\}$

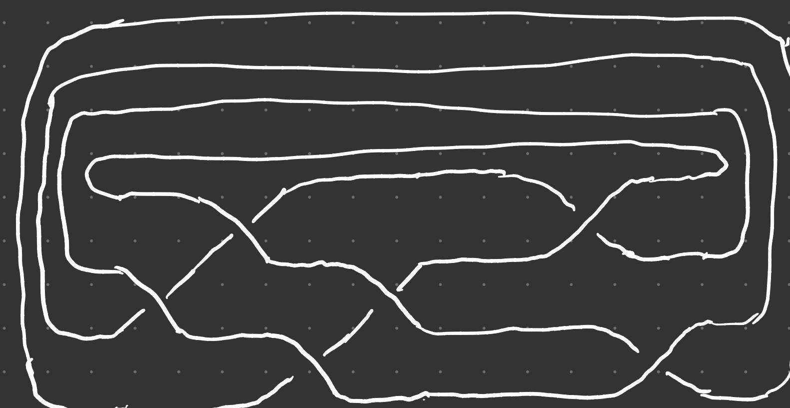
Theorem [Galashin–L.]

$$\text{mixed Hodge polynomial}\left(H^*_T(\mathring{\Pi}_f)\right) = \text{top "a"-coefficient of Khovanov–Rozansky triply graded link homology for } \hat{\beta}_f$$

$f \longrightarrow (w = s_2 s_1 s_3 s_2,\ v = s_3 s_1)$

$\beta_f =$ 

$\parallel$



$$\widehat{\beta\,\mathrm{id}_{k,n}} = \hat{\beta}_{k,n} = (k, n-k)\text{-torus link}$$

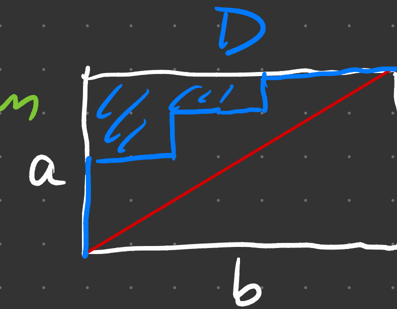
$\hat{\beta}_{2,4} =$  $\hat{\beta}_{2,5} =$  $\hat{\beta}_{3,7} =$ 

$\gcd(k,n) = 1$ $\quad X_{k,n} := \mathring{\Pi}_{\mathrm{id}_{k,n}} / T$ $\quad$ Catalan variety

Theorem [Galashin–L.]

mixed Hodge polynomial $(X_{k,n}) = C_{k,n-k}(q,t)$ $\quad$ q,t-rational Catalan number

$$C_{a,b} = \frac{1}{a+b}\binom{a+b}{a}$$ rational Catalan number



$$C_{n,n+1} = C_n$$ Catalan

$$C_n(q,1) = \sum_D q^{\text{area}(D)} \qquad C_n\left(q,\frac{1}{q}\right) \doteq \frac{1}{[a+b]_q}\begin{bmatrix}a+b\\ a\end{bmatrix}_q$$

Theorem [Galashin–L.]

$$\# X_{k,n}(\mathbb{F}_q) = \frac{1}{[n]_q}\begin{bmatrix}n\\ k\end{bmatrix}_q$$

$$\text{Poincare poly}(X_{k,n}) = C_{k,n-k}(q^2,1)$$ cohomology is even!

Mixed Hodge numbers

Deligne's mixed Hodge structure $\Longrightarrow$

$$H^k(X) \cong \bigoplus_{p,q} H^{k,(p,q)}$$

$\mathring{\Pi}_{k,n}$ is of mixed-Tate (or Hodge-Tate) type.

$$H^k(X) \cong \bigoplus_{\frac{k}{2} \leq p \leq k} H^{k,(p,p)}$$

(smooth)

No Lefschetz theorem, but we have

Haiman, Gorsky-Negut, Mellit

Theorem [L.-Speyer (+ Galashin-L.)]

$\mathring{\Pi}_f$ satisfies curious Lefschetz $\Rightarrow$ $C_{k,n-k}(q,t) = C_{k,n-k}(t,q)$ + unimodal

	H^0	H^1	H^2	H^3	H^4	H^5	H^6	H^7	H^8
$k-p=0$	1	0	1	0	1	0	1	0	1
$k-p=1$					1	0	1		

$H^{4,(4,4)}$

$H^{4,(3,3)}$

symmetry line.

mixed Hodge table for $X_{3,8}$

$\dim X_{3,8} = 8 = 3\times 5 - 7$

$$\#Gr(k,n)(\mathbb{F}_q) = \begin{bmatrix} n \\ k \end{bmatrix}_q$$

Gaussian polynomial

<u>Corollary</u>

$$\mathrm{Prob}\left(V \in \mathring{\Pi}_{id_{k,n}}\right) = \frac{(q-1)^n}{(q^n-1)}$$

Homfly polynomial

L_+ L_- L_0

$P(K) \in \mathbb{C}[a^{\pm 1}, z^{\pm 1}]$

$$a\, P(L_+) - a^{-1} P(L_-) = z\, P(L_0)$$

$$P(\bigcirc) = 1$$

Theorem [Galashin–L.]

$$\#\mathring{\Pi}_{k,n}(\mathbb{F}_q) = (q-1)^{n-1}\, P^{top}\big((k,n-k)\text{-torus knot}\big)\Big|_{\substack{a=q^{-1/2} \\ z=q^{1/2}-q^{-1/2}}}$$

e.g. $\mathring{\Pi}_{2,5} \longrightarrow$  $P(a,z) = a^{-2}(z^2+2) - a^{-4} \rightsquigarrow 1+q^2$

Homfly $\rightsquigarrow$ Kazhdan-Lusztig R-poly.