Puzzles compute the Euler characteristic of the intersection of Bruhat cells

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March 20, 2021

Abstract

In 2017, we showed that 1-, 2-, and 3-step Schubert calculus puzzles could be derived as a $q \to 0$ limit of an A_2, D_4, E_6 quantum integrable system (respectively). It turns out that the $q \neq 0$ systems compute a richer product of "Segre-Schwartz-MacPherson" classes most naturally defined not on those flag manifolds, but on their cotangent bundles.

As a consequence, we obtain (again for up to 3-step) a positive formula for the Euler characteristic of the intersection of three generically situated Bruhat cells (times the usual K-theoretic sign, although, this is not a K-theoretic calculation).

We conjecture that this signed positivity holds for general G/P.

The Euler characteristic statement.

Theorem [K-ZJ '21]. Let $F^\bullet_{1,2,3}$ be three flags in $\mathbb{C}^n$ in generic relative position, and $X^\circ_\lambda(F^\bullet_1)$, $X^\circ_\mu(F^\bullet_2)$, $X^\circ_\nu(F^\bullet_3)$ be three Bruhat cells (not their closures!) in G/P. Let

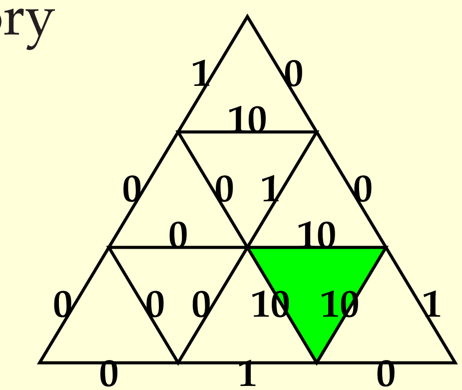
$$d := \dim(G/P) - \ell(\lambda) - \ell(\mu) - \ell(\nu) = \dim\left(X^\circ_\lambda(F^\bullet_1) \cap X^\circ_\mu(F^\bullet_2) \cap X^\circ_\nu(F^\bullet_3)\right)$$

Then $(-1)^d \chi\left(X^\circ_\lambda(F^\bullet_1) \cap X^\circ_\mu(F^\bullet_2) \cap X^\circ_\nu(F^\bullet_3)\right) \geq 0$ if G/P is a 1-, 2-, or 3-step flag manifold, in that it counts a simple set of puzzles. For 4-step, we have a nonpositive puzzle formula.

We conjecture this alternating-sign property holds for general G/P.

In the Grassmannian case, the relevant puzzle pieces are the usual ones, plus the 10-10-10 Δ **and** ∇ that previously came up in K-theory (as Δ only, for structure sheaves; ∇ only, for ideal sheaves).

Example: $G/P = \mathbb{CP}^2$, $\nu = 010$ defining a $\mathbb{CP}^1$ minus a point, $\mu = \nu = 001$ each excising another: $\bigcap \cong \mathbb{CP}^1 \setminus$ three points. That has $d = 1$ and $\chi = \chi(\mathbb{CP}^1) - 3 = -1$, and indeed there is a unique puzzle.



In the Grassmannian case, each puzzle uses exactly d 10-10-10 pieces, so instead of counting one could let them contribute factors of -1 as in K-theory.

Plan of the talk:
how to do computations of this sort, cohomologically.

(Such computations weren't our goal; they are just a side-effect.)

Plan for this talk (with $\star$ marking our results):

1. Use $\mathcal{D}$-modules to associate a class to a locally closed submanifold $A \subseteq M$.
2. Show this association has an additivity property, much like χ does, and when $A \pitchfork B$ a multiplicativity property (due to Schürmann).
3. In the case $M = G/B$, relate this construction to representation theory. (We won't really need this relation at all.)
4. Use the Grothendieck-Springer degeneration $G/T \rightsquigarrow T^*G/B$ to see a recurrence relation on these classes.
5. $\star$ State our puzzle formula for multiplying (these classes)/[zero section] in $\widetilde{H}^*_{T\times\mathbb{C}^\times}(Gr(k, \mathbb{C}^n))$ (with hints about d-step flag manifolds, $d \leq 4$).
6. $\star$ Obtain the χ computation as the "nonequivariant" case thereof.

We don't need #4 directly in this talk, but it is a key step in proving #5 (which we won't). Principally this talk is to motivate the definition of #5's SSM classes.

Some sheaves of algebras and modules over M.

Let $Vec(M)$ denote the sheaf of vector fields on M. This acts on $\mathcal{O}_M$, the sheaf of functions, by directional derivatives.

There are two ways to multiply vector fields, **the second the gr of the first:**

1. (noncommutative, filtered) Compose the operators, obtaining higher-order differential operators, living in the sheaf $\mathcal{D}_M$.
2. (commutative, graded) Regard a section of $T(M)$ as a fiberwise-linear function on $T^*(M)$, and multiply these to obtain higher-degree functions, living in the sheaf $\mathcal{O}_{T^*M}$.

For $\iota : V \hookrightarrow V \oplus W$ linear, let $\iota_*(\mathcal{O}_V)$ be polynomials $\sum \overbrace{v_1 v_2 \cdots}^{\text{coordinates}} \overbrace{\delta_{w_1}\delta_{w_2}\cdots}^{\text{Dirac-deltas}}$, which we then know how to differentiate w.r.t. vector fields on $V \oplus W$. This recipe extends to $\iota : A \hookrightarrow M$ a locally closed submanifold.

Given a compatible filtration on $\iota_*(\mathcal{O}_A)$, its $\mathrm{gr}(\iota_*(\mathcal{O}_A))$ becomes a $\mathcal{O}_{T^*M}$-module. Luckily, "good filtrations" exist uniquely enough for our purposes of obtaining a **characteristic cycle** $cc(1_A)$ on T^*M, the support of $\mathrm{gr}(\iota_*(\mathcal{O}_A))$.

This $cc(1_A)$ will be the conormal bundle to A, plus an $\mathbb{N}$-combination of conormal varieties over $\overline{A} \setminus A$.

Chern-Schwartz-MacPherson classes and some properties.

Consider $[cc(1_A)] \in H^*_{\mathbb{C}^\times}(T^*M) \cong H^*_{\mathbb{C}^\times}(M) \cong H^*(M) \otimes H^*_{\mathbb{C}^\times} \cong H^*(M)[\hbar]$.

If $B \subseteq A$ is a closed hypersurface, then a certain short exact sequence of $\mathcal{D}_M$-modules implies $[cc(1_A)] = [cc(1_{A \setminus B})] - [cc(1_B)]$.
So we set $csm(1_A) := (-1)^{\mathrm{codim} A}[cc(1_A)]$ to absorb the signs and be simply additive ["Ginsburg" 1987]. In particular now one can safely define $csm(f)$ for any constructible function f on M.

This recipe satisfies three key properties (though we only use the latter two):

1. If $\pi : X \to Y$ is a proper map, and $A \subseteq X$ is locally closed, then $\pi_*(csm(1_A)) = csm(1_{\pi_*(f)})$ where $(\pi_*(f))(y) := \chi(A \cap f^{-1}(y))$.
2. If $A = M$, then $csm(1_A) = e(T^*A)$, the equivariant Euler class.
3. [Schürmann '17] If $A, B \subseteq M$ are smooth and locally closed, and their (singular) closures $\overline{A}, \overline{B}$ intersect stratified-transversely, then $csm(1_A) \cup csm(1_B) = csm(1_{A \cap B}) \cup e(TM)$.

Deligne-Grothendieck observed that #1 and #2 make csm unique; MacPherson '74 proved that it exists, but not as directly as Ginzburg's construction does. (They also dehomogenize $\hbar$, to -1, which is just silly.)

The special case of $M = G/B$, part I.

Since $G \circlearrowright G/B$, we have $\mathfrak{g} \to \Gamma(\mathrm{Vec}(G/B))$, and $U\mathfrak{g} \to \Gamma(D_{G/B})$. In particular, any $\mathcal{D}_{G/B}$-module $\mathcal{F}$ gives a $U\mathfrak{g}$-representation $\Gamma(\mathcal{F})$.

Theorem [Beilinson-Bernstein].

1. This $U\mathfrak{g} \to \Gamma(D_{G/B})$ is onto, with kernel generated by $Z(U\mathfrak{g}) \cap \ker(U\mathfrak{g} \circlearrowright \mathbb{C}_{triv})$. Let $(U\mathfrak{g})_0$ denote this "trivial central character" quotient.
2. The functor $\mathcal{F} \mapsto \Gamma(\mathcal{F})$ defines an equivalence of categories between $\mathcal{D}_{G/B}$-modules and $(U\mathfrak{g})_0$-representations.

Main example. Let $\iota_*(\mathcal{O}_{X_w^\circ})$ be the $\mathcal{D}_{G/B}$-module associated to the B_- orbit through wB/B. Then its representation is a Verma module (for B_-) and its characteristic cycle computes $csm(1_{X_w^\circ})$.

More generally, if $U\mathfrak{g} \circlearrowright V$ is any representation in category $\mathcal{O}$, which in particular means that B_- acts (compatibly with $U\mathfrak{b}_-$), then the characteristic cycle is a union of conormal varieties to Schubert varieties.

We will need two (independent) variations: G/P, and central characters $\varepsilon \neq 0$, which require "twisted" $\mathcal{D}$-modules we'll call $\mathcal{D}^\varepsilon$-modules.

The special case of $M = G/B$, part II.

The **Grothendieck-Springer deformation** of T^*G/B (here $G = GL_n$) is triples

$$\{(X \in \mathfrak{g},\ F^\bullet \in F\ell(\mathbb{C}^n),\ \varepsilon \in \mathbb{C}^n)\ :\ (X - \varepsilon_i)F_i \leq F_{i-1}\}$$

For any fixed ε, this is a bundle over G/B:

- at $\varepsilon = 0$ it's Springer's picture of T^*G/B, whereas
- at generic ε it's $G \cdot diag(\varepsilon) \cong G/T$.

The B_--orbits $B_- wB/T \subseteq G/T$ are **all closed** and are **transitively permuted** by the (perfectly honest) right W-action on G/T.

That right W-action doesn't honestly extend to $\varepsilon = 0$, but we can replace each $\cdot r_\alpha$ by its graph and take the $\varepsilon \to 0$ limit of the graph, as a correspondence.

Loose picture

$$\begin{array}{ccc} \mathcal{D}^\epsilon & \rightsquigarrow & \mathcal{D} \\ \wr & & \wr \\ \mathcal{O}_{G/T} & \rightsquigarrow & \mathcal{O}_{T^*G/B} \end{array}$$

algebras on G/B

$$\begin{array}{ccc} \text{irred Verma} & \rightsquigarrow & \varepsilon{=}0 \text{ Verma} \\ \wr & & \wr \\ \mathcal{O}_{B_- wT/T} & \rightsquigarrow & \mathcal{O}_{cc(1_{X_w^\circ})} \end{array}$$

modules on G/B

This is what leads to the operators $r_\alpha + \hbar\partial_\alpha$ that play the role for $\{csm(X_w^\circ)\}$ of the divided difference operators: $(r_\alpha + \hbar\partial_\alpha) \cdot csm(X_w^\circ) = csm(X_{wr_\alpha}^\circ)$, $\forall w, \alpha$.

Puzzles for multiplying Segre-Schwarz-MacPherson classes.

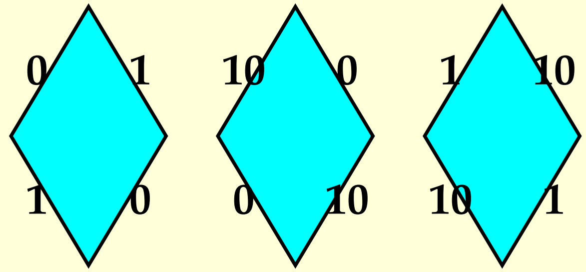
The sets $\{[cc(1_{X_w^\circ})]\}$, $\{[cc(1_{X^w_\circ})]\}$ are dual bases w.r.t. integration on T^*G/P (definable only through equivariant localization, since it's noncompact).
Trying to exploit this down on G/P leads one to the **SSM classes**

$$ssm(1_{X_w^\circ}) := csm(1_{X_w^\circ}) \,/\, e(T^*G/P) \qquad \in \text{ localized } H^0_{T\times\mathbb{C}^\times}(G/P).$$

Theorem [K-ZJ '21].

For $G/P = Gr(k, \mathbb{C}^n)$, the ssm structure constants

1. can be computed as a sum over the same puzzles as on p1, plus *three* equivariant pieces:



2. There are similar puzzle rules for d-step flag manifolds, $d \leq 4$ where the edge labels index bases of certain small representations of A_2, D_4, E_6, E_8 respectively. The (fugacities of the) puzzle pieces correspond to the nonzero entries (*anything* weight-preserving) in the corresponding rational R-matrix. (For $d > 1$ we make use of quiver varieties that aren't cotangent bundles!)
3. For $d = 1, 2$ it is straightforward to take $\hbar \to \infty$ to recover puzzle rules for Schubert classes. For $d = 3$ one can control the infinities only after losing equivariance. For $d = 4$ the resulting rule is not positive.
4. There are K-theory versions of all these, with exactly the same puzzles, but using the entries in the corresponding trigonometric R-matrix.