Tangent spaces and T-stable curves of Schubert varieties

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Curves and Tangent Spaces

Setup:

- X = a Schubert variety in the generalized flag variety G/P
- x = a T-fixed point of X.

We study two spaces:

- $T_x(X)$ = tangent space to X at x
- $TE_x(X)$ = span of tangent lines to T-invariant curves through x

These spaces are characterized by their weights: Φ_{tan} and Φ_{cur} respectively.

Known:

- $\Phi_{\mathrm{cur}} \subseteq \Phi_{\mathrm{tan}}$
- Equality in type A.

Curves and Tangent Spaces

- The set Φ_{cur} of curve weights is relatively easy to understand.
  - Φ_{cur} only depends on the Weyl group
- The tangent space weights Φ_{tan} is more difficult.
  - Depends on the root system, not just the Weyl group
  - Described for classical groups (Lakshmibai, Seshadri): complicated
  - No uniform description
  - Not known in exceptional types

Curves and Tangent Spaces

Main Result

The main result of this talk proves a relation between the sets Φ_{tan} and Φ_{cur}.

If R is a subset of a vector space, let $\text{Cone}_A R$ denote the set of non-negative linear combinations of elements of R, with coefficients in the ring A. In this talk, A will be either $\mathbf{Z}$ or $\mathbf{Q}$.

Theorem

Suppose G is of classical type.

$$\Phi_{\text{tan}} \subseteq \text{Cone}_A \Phi_{\text{cur}}.$$

- This theorem holds with $A = \mathbf{Q}$ in all types; for simply laced types it also holds with $A = \mathbf{Z}$.
- Expected to be true in exceptional types, but part of the argument involves a case by case check.

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Equivalent formulations:

- Φ_{tan} and Φ_{cur} generate the same cone over A.
- Φ_{tan} and Φ_{cur} have the same A-indecomposable elements.

In certain situations, one can prove more.

Theorem

Suppose that G is simply laced and that either

1. *G/P is cominuscule*
2. *x is a cominuscule Weyl group element.*

Then $\Phi_{\mathrm{tan}} = \Phi_{\mathrm{cur}}$.

Notation

To explain these results we need a little notation.

- G = simple algebraic group
- B = Borel subgroup, B^- = opposite Borel subgroup
- T = maximal torus contained in B
- $B = TU$, $B^- = TU^-$
- $X = G/B$, the flag variety
- W = Weyl group, equipped with Bruhat order
- The T-fixed points of X are xB for $x \in W$. Often write x for xB.

More Notation

- Schubert variety $X^w = \overline{B^- \cdot wB} \subset X$
- T-fixed points in X^w are the xB with $x \geq w$
- Kazhdan-Lusztig variety $Y_x^w = X^w \cap UxB$.
- Near x, X^w looks like the product of Y_x^w and a representation of T, so the results of the paper are proved by studying Y_x^w
- Using the Kazhdan-Lusztig variety in place of the Schubert variety, define tangent and curve weights $\Phi_{\text{tan}}^{\text{KL}}$ and $\Phi_{\text{cur}}^{\text{KL}}$.
- The main result follows from the stronger statement

$$\Phi_{\text{tan}}^{\text{KL}} \subseteq \text{Cone}_A \, \Phi_{\text{cur}}^{\text{KL}}.$$

The 0-Hecke algebra

This result relies on equivariant K-theory and the 0-Hecke algebra.

Let $\hat{T} = \mathrm{Hom}(T, \mathbf{G}_m)$. The representation ring $R(T)$ is the ring spanned by e^λ for $\lambda \in \hat{T}$, with multiplication $e^\lambda e^\mu = e^{\lambda+\mu}$.

Definition

The 0-Hecke algebra is a free $R(T)$-algebra with basis H_u, for $u \in W$. Multiplication: Let s be a simple reflection.

- $H_s H_u = H_{su}$ if $l(su) > l(u)$
- $H_s H_u = H_u$ if $l(su) < l(u)$
- $H_s^2 = H_s$
- H_1 is the identity element.

The Demazure product and inversion sets

Suppose $\mathbf{s} = (s_1, s_2, \ldots, s_l)$, where $s_i = s_{\alpha_i}$ is a simple reflection. This expression need not be reduced.

- The Demazure product $z_{\mathbf{s}} \in W$ is defined by the formula

$$H_{s_1} \cdots H_{s_l} = H_{z_{\mathbf{s}}}.$$

- We have $z_{\mathbf{s}} \geq s_1 s_2 \cdots s_l$ with equality if $\mathbf{s}$ is reduced.

Now suppose $\mathbf{s}$ is a reduced expression for $x \in W$.

- Define $\gamma_1 = \alpha_1,\ \gamma_2 = s_1(\alpha_2),\ \gamma_3 = s_1 s_2(\alpha_3), \ldots$.
- Then $I(x^{-1}) = \{\gamma_1, \gamma_2, \ldots, \gamma_\ell\}$.

For the rest of this talk we will fix x and the reduced expression $\mathbf{s}$. We denote $I(x^{-1})$ by S.

Weights on inversion sets

Recall that $S = \{\gamma_1, \gamma_2, \ldots, \gamma_\ell\}$. Let $\mathbf{s}_i$ denote the sequence obtained by deleting the reflection s_i from $\mathbf{s}$. Define two maps $z, x : S \to W$ by the rule

$$z(\gamma_i) = z_{\mathbf{s}_i}$$

and

$$x(\gamma_i) = s_1 s_2 \cdots \hat{s}_i \cdots s_\ell.$$

Write

$$z_i = z(\gamma_i),\ x_i = x(\gamma_i).$$

Given $w \in W$, we define

$$S_{z \geq w} = \{\gamma_i \in S \mid z_i \geq w\}$$

and

$$S_{x \geq w} = \{\gamma_i \in S \mid x_i \geq w\}.$$

Curve weights and tangent weights

Carrell-Peterson proved that

$$\Phi^{\mathrm{KL}}_{\mathrm{cur}} = S_{x \geq w}$$

The connection with tangent spaces is due to the following result:

Theorem

$$\Phi^{\mathrm{KL}}_{\tan} \subseteq \mathrm{Cone}_{\mathbf{Z}}(S_{z \geq w}).$$

Relating $S_{z \geq w}$ and $S_{x \geq w}$

Our main result is a consequence of the following theorem.

Theorem

$\operatorname{Cone}_A S_{z \geq w} = \operatorname{Cone}_A S_{x \geq w}$.

- Since $z_i \geq x_i$, the inclusion $\operatorname{Cone}_A S_{z \geq w} \supseteq \operatorname{Cone}_A S_{x \geq w}$ is immediate.
- The reverse inclusion is proved by introducing some notions of decomposability of weights and relating them.

Decomposability

Definition

1. A linear combination $\alpha = \sum c_i \alpha_i$, with $c_i \in A$ is said to be A-increasing if each $z(\alpha_i) \geq z(\alpha)$.
2. An **iso-decomposition** is a $\mathbf{Q}$-decomposition of the form $\alpha = c\alpha_1 + c\alpha_2$ with $\|\alpha_1\| = \|\alpha_2\|$.

An element without one of these decompositions (relative to some fixed set of roots) is called increasing A-indecomposable or iso-indecomposable, respectively.

Decomposability

The outline of the proof is as follows.

1. Every element of $S_{z \geq w}$ can be written as an increasing A-linear combination of increasing A indecomposable elements which lie in $S_{z \geq w}$.
2. Increasing A indecomposable elements are iso-indecomposable.
3. Iso-indecomposable elements γ_i satisfy $z_i = x_i$.

Hence:

- Every element of $S_{z \geq w}$ is a positive A linear combination of elements which lie in $S_{x \geq w}$.
- Thus $S_{z \geq w} \subseteq \mathrm{Cone}_A \, S_{x \geq w}$, completing the proof.

Comments on the proof

The fact that every element of S can be decomposed into indecomposables requires more effort than might be expected because we are using $\mathbf{Q}$ coefficients, and so a naive approach to decomposing may not terminate.

Comments on the proof

Example

Suppose Φ is of type B_2, $S = \Phi^+ = \{\epsilon_1, \epsilon_2, \epsilon_1 - \epsilon_2, \epsilon_1 + \epsilon_2\}$. The element ϵ_1 has a rational decomposition

$$\epsilon_1 = \frac{1}{2}(\epsilon_1 + \epsilon_2) + \frac{1}{2}(\epsilon_1 - \epsilon_2).$$

The long root $\epsilon_1 + \epsilon_2$ has a rational decomposition as a sum of the short roots ϵ_1 and ϵ_2. $\epsilon_1 + \epsilon_2 = (\epsilon_1) + (\epsilon_2)$. Decompose the summand ϵ_1 as above:

$$\epsilon_1 = \frac{1}{2}\Big(\frac{1}{2}(\epsilon_1 + \epsilon_2) + \frac{1}{2}(\epsilon_1 - \epsilon_2) + \epsilon_2\Big) + \frac{1}{2}(\epsilon_1 - \epsilon_2).$$

This process can be repeated indefinitely without terminating. In this example, the indecomposables are $\{\epsilon_1 - \epsilon_2, \epsilon_2\}$, and $\epsilon_1 = (\epsilon_1 - \epsilon_2) + \epsilon_2$ is the desired **Q** decomposition of ϵ_1 by indecomposables.

Iso-indecomposability

Iso-indecomposability enters into the picture as follows. Recall that $\mathbf{s} = (s_1, s_2, \ldots, s_l)$ and $\mathbf{s}_i$ is the sequence obtained by deleting the reflection s_i from $\mathbf{s}$. z_i is the Demazure product of $\mathbf{s}_i$ and x_i is the product of the elements in $\mathbf{s}_i$.

- If $\mathbf{s}_i$ is reduced, then $z_i = x_i$.
- If $\mathbf{s}_i$ is not reduced, then γ_i is iso-decomposable.
- More precisely, one can find $j < i < k$ such that $|\gamma_j| = |\gamma_i|$ and
$$c\gamma_i = \gamma_j + \gamma_k.$$
- Moreover, $z_j \geq z_i$ and $z_k \geq z_i$ so this decomposition is increasing.