Positivity in Peterson Schubert Calculus

Joint work with Leonardo Mihalcea and Rahul Singh

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AMS Special Session on Schubert Calculus

Cast of characters

G complex semi-simple Lie group, with Lie algebra $\mathfrak{g}$
B choice of Borel, with Lie algebra $\mathfrak{b}$
B^{-} opposite Borel
$T = B \cap B^{-}$ a maximal torus
Δ set of positive simple roots for G.
S circle subgroup of T, generated by $h \in L(T)$ with $\alpha(h) = 2$ for all $\alpha \in \Delta$
$t = \alpha|_S$
$\mathfrak{g}_\alpha$ root space of α

Definition (Peterson variety)

Let $e \in \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$ be a principal nilpotent element. A Peterson variety may be given by

$$\mathbf{P} = \{gB \in G/B : Ad(g^{-1})e \in \mathfrak{b} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_{-\alpha}\}$$

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Peterson variety

Type A Peterson

Let N be the matrix whose Jordan canonical form consists of one block with 1's on the superdiagonal and 0's elsewhere. Then $\mathbf{P} \subseteq Fl(n;\mathbb{C})$ is the collection of flags over $\mathbb{C}^n$ satisfying $NV_i \subseteq V_{i+1}$.

Theorem (Tymoczko, Precup)

P *has a paving by affines.*

Example (Cells for type A, n=3)

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cup \begin{pmatrix} a & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cup \begin{pmatrix} 1 & 0 & 0 \\ 0 & b & 1 \\ 0 & 1 & 0 \end{pmatrix} \cup \begin{pmatrix} c & d & 1 \\ d & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Theorem (G.-Mihalcea-Singh)

The cell closures of this affine paving are smaller Peterson varieties.

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Peterson variety fixed points

Fixed points

$$\mathbf{P}^S \leftrightarrow \{w_I : I \subset \Delta\}$$

Weyl group elements realizable as the longest words w_I for subsets I of the simple roots.

Example

For $n = 3$ in type A, the fixed points are

$$\begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}, \begin{pmatrix}0 & 1 & 0\\1 & 0 & 0\\0 & 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 0 & 0\\0 & 0 & 1\\0 & 1 & 0\end{pmatrix}, \begin{pmatrix}0 & 0 & 1\\0 & 1 & 0\\1 & 0 & 0\end{pmatrix}$$

Primer on Equivariant Cohomology I

Schubert varieties $X_u = \overline{BuB}/B$ for $u \in W$. Then there is a dual relationship

$$\{[X_u]\}_{u \in W} \rightsquigarrow \{\sigma_u\}_{u \in W}$$

where $\sigma_u := [X^u]$.
$H_T^*(G/B)$ is a free module over $H_T^*(pt)$, with basis $\{\sigma_w\}$. Therefore,

$$\sigma_u \cdot \sigma_v = \sum_w c_{u,v}^w \sigma_w$$

for some $c_{u,v}^w \in H_T^*(pt)$.

Theorem (Graham, 1999)

The polynomials $c_{u,v}^w \in H_T^*(pt)$ are polynomials in the simple roots with nonnegative coefficients.

A basis for $H_S^*(\mathbf{P})$

Theorem (Harada-Tymoczko, Drellich, G-Mihalcea-Singh)

Pick a subset $A \subset \Delta$. For each A, choose a Coxeter element v_A i.e. a product $s_{\alpha_1} \cdots s_{\alpha_k}$ over all elements $\alpha_i \in A$. Define

$$p_A = i^*(\sigma_{v_A}),$$

where σ_{v_A} is the S-equivariant Schubert class obtained by restricting the T action to S. Then $H_S^*(\mathbf{P})$ is a free module over $H_S^*(pt)$ with basis $\{p_A\}$.

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$$p_A \cdot p_B = \sum_C b_{A,B}^C p_C$$

defines equivariant constants $b_{A,B}^C \in H_S^*(pt) \cong \mathbb{C}[t]$.

Remark. We write p_A but it depends on v_A.

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POSITIVITY

THEOREM (G.-MIHALCEA-SINGH)

The structure constants $b_{A,B}^{C} \in H_S^*(pt)$ defined by

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are polynomials in t with non-negative coefficients.

Before we talk about why this is true, a brief interlude...

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A positive! Integral! formula in type A

Structure constants $b^C_{A,B}$ defined by

$$p_A p_B = \sum_C b^C_{A,B}\, p_C$$

Let $\mathcal{T}_A = min(A)$, $\mathcal{H}_A = max(A)$

Theorem (G.-Gorbutt)

Let $A, B, C \subset \{1, \ldots, n-1\}$ be nonempty and consecutive. If $A \cup B \subseteq C$, and $d := |A| + |B| - |C| \geq 0$, then

$$b^C_{A,B} = d! \binom{\mathcal{H}_A - \mathcal{T}_B + 1}{d,\ \mathcal{T}_A - \mathcal{T}_C,\ \mathcal{H}_C - \mathcal{H}_B} \binom{\mathcal{H}_B - \mathcal{T}_A + 1}{d,\ \mathcal{T}_B - \mathcal{T}_C,\ \mathcal{H}_C - \mathcal{H}_A} t^d.$$

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A POSITIVE! INTEGRAL! FORMULA IN TYPE A

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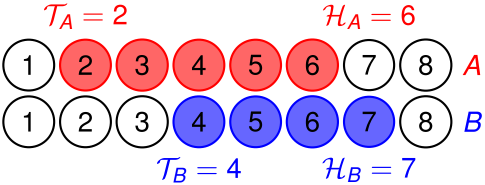

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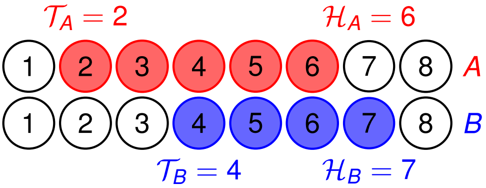

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WHERE $d = |A| + |B| - |C|$.

EXAMPLE

Let $A = \{1, 2\}$, $B = \{2, 3\}$ and $C = \{1, 2, 3\}$, so $d = 1$.

$\mathcal{T}_A = 1 \quad \mathcal{H}_A = 2$

$\mathcal{T}_C = 1, \mathcal{H}_C = 3$

$\max(\mathcal{T}_A, \mathcal{T}_B) = 2$

$\mathcal{T}_B = 2 \qquad \mathcal{H}_B = 3$

$\min(\mathcal{H}_A, \mathcal{H}_B) = 2$

$$b_{12,23}^{123} = 1!\binom{2-2+1}{1,\ 1-1,\ 3-3}\binom{3-1+1}{1,\ 2-1,\ 3-2} t = \binom{1}{1}\frac{3!}{1!1!1!} = 6t.$$

Similarly, $b_{12,23}^{1234} = 3$. All other $b_{12,23}^{C} = 0$. Thus $p_{12}p_{23} = (6t)p_{123} + 3p_{1234}$.

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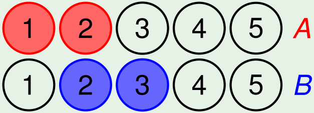

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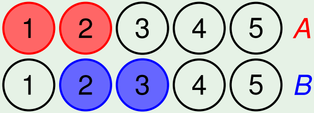

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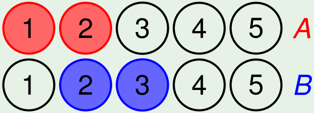

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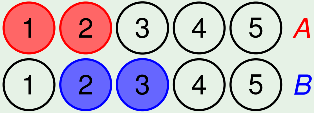

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Why is Peterson Schubert calculus positive?

$$p_A p_B = \sum_C b_{A,B}^C p_C, \quad p_A := \iota^*(\sigma_{v_A})$$

Define Peterson Schubert varieties

$$\mathbf{P}_A = \overline{\mathbf{P} \cap Bw_A B/B}$$

These are subvarieties of **P** but also subvarieties of G/B.
We would like a dual relationship

$$\{[\mathbf{P}_A]\}_{A\subset\Delta} \rightsquigarrow \{p_A\}_{I\subset\Delta}$$

THEOREM (DUALITY) (G.-MIHALCEA-SINGH)

Let A, B be subsets of the set of simple roots Δ and $v_A \in W$ a Coxeter element for A. Then

$$\langle p_A, [\mathbf{P}_B]_S \rangle = \delta_{A,B} m(v_A),$$

where $m(v_A) > 0$ is an integer.

LEMMA

The intersection $X^{v_A} \cap \mathbf{P}_A$ is a single point, w_A.

Proof. The intersection is S invariant, and contains the point w_A. Fixed points of $\mathbf{P}_A$ are of the form w_B for some B with $B \subset A$. Fixed points of X^{v_A} are of the form $v \geq v_A$. But $w_B \geq w_A$ implies $A \subset B$. Since it's the only fixed point in a stable invariant space, the intersection is the point. □

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The number $m(v_A)$ is the multiplicity of the intersection.

Diagram	$m(s_1 \cdots s_n)$	Diagram	$m(s_1 \cdots s_n)$
A_n	1	F_4	48
B_n, C_n	2^{n-1}	E_6	72
D_n	2^{n-2}	E_7	864
G_2	6	E_8	51840

TABLE: Values for the pairing $\langle p_A, [\mathbf{P}_A] \rangle$.

Peterson Schubert calculus

- Schubert calculus is positive

$$\sigma_u \cdot \sigma_v = \sum_w c_{u,v}^w \sigma_w, \quad c_{u,v}^w \in H_S^*(pt)$$

- $\implies p_A \cdot p_B = \iota^*(\sigma_{v_A}) \cdot \iota^*(\sigma_{v_B}) = \iota^*(\sigma_{v_A} \cdot \sigma_{v_B}) = \sum_w c_{v_A,v_B}^w \iota^* \sigma_w$
- Thus positivity follows from showing positivity of coefficients here:

$$\iota^*(\sigma_w) = \sum_A b_u^A p_A$$

- Duality theorem implies that b_u^A are positive if and only if the push forwards in homology are positive:

$$\iota_*(\mathbf{P}_A) = \sum_u c_A^u [X_u], \quad c_A^u \geq 0.$$

- Graham positivity $\implies \iota_* \mathbf{P}_A = \sum_u c_A^u [X_u]$ with c_A^u polynomials in t with nonnegative coefficients.

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$$\sigma_u \cdot \sigma_v = \sum_w c_{u,v}^w \sigma_w, \quad c_{u,v}^w \in H_S^*(pt)$$

- $\implies p_A \cdot p_B = \iota^*(\sigma_{v_A}) \cdot \iota^*(\sigma_{v_B}) = \iota^*(\sigma_{v_A} \cdot \sigma_{v_B}) = \sum_w c_{v_A,v_B}^w \iota^* \sigma_w$
- Thus positivity follows from showing positivity of coefficients here:

$$\iota^*(\sigma_w) = \sum_A b_u^A p_A$$

- Duality theorem implies that b_u^A are positive if and only if the push forwards in homology are positive:

$$\iota_*(\mathbf{P}_A) = \sum_u c_A^u [X_u], \quad c_A^u \geq 0.$$

- Graham positivity $\implies \iota_* \mathbf{P}_A = \sum_u c_A^u [X_u]$ with c_A^u polynomials in t with nonnegative coefficients.

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Theorem (Graham Positivity)

Let B' be a connected solvable group with unipotent radical U', and let $T' \subset B'$ be a maximal torus, so that $B' = T'U'$. Let $\alpha_1, \ldots, \alpha_d$ be the weights of T' acting on $\mathrm{Lie}(U')$. Let X be a scheme with a B'-action, and Y a T'-stable subvariety of X. Then there exist B'-stable subvarieties $D_1, \ldots, D_k$ of X such that in the equivariant homology $H_*^{T'}(X)$,

$$[Y]_{T'} = \sum f_i [D_i]_{T'},$$

where each $f_i \in H^*_{T'}(pt)$ is a linear combination of monomials in $\alpha_1, \ldots, \alpha_d$ with non-negative integer coefficients.

Application:

- Let $X = G/B$, $Y = \mathbf{P}_I$, $T' = S$, and $B' = SU$. The B'-stable varieties are Schubert varieties $\{[X_u]\}$. Then $\iota_*(\mathbf{P}_A) = \sum c^u_A [X_u]$ with c^u_A a positive polynomial in t.

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Easy proof of positivity!

- The projection formula implies

$$\langle \iota^*(\sigma_w), [P_B] \rangle = \langle \sigma_w, \iota_*[P_B] \rangle = \langle \sigma_w, \sum_u c_B^u [X_u] \rangle = c_B^w.$$

- c_B^w polynomials in t with nonnegative coefficients
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$$\langle \iota^*(\sigma_w), [P_B] \rangle = \sum_A b_w^A \langle p_A, [P_B] \rangle = b_w^B m(v_B)$$

  Thus $m(v_B) b_w^B = c_B^w$.
- Geometry implies $m(v_A) > 0$.
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- Finally

$$p_A p_B = \sum_w c_{v_A, v_B}^w \iota^*(\sigma_w) = \sum_{w,C} c_{v_A, v_B}^w b_w^C p_C$$

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Remarks on cellular structure and stability

- Peterson varieties have a cellular structure, given by the closures of *cells*

$$\mathbf{P}_A^\circ = \mathbf{P} \cap Bw_A B/B.$$

Theorem (Stability, G-Mihalcea-Singh)

Consider the natural inclusion $i : G_A/B_A \hookrightarrow G/B$.

1. *The Peterson associated to A in G_A/B_A includes into G/B as the intersection with $\mathbf{P}$.*
2. *For $B \subset A$, we have $i_*([\mathbf{P}_B]_S) = [\mathbf{P}_B]_S$.*
3. *For $B \subset \Delta$, we have* $i^*(p_B) = \begin{cases} p_B & \text{if } B \subset A, \\ 0 & \text{otherwise.} \end{cases}$

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Thank you!