Equivariant coherent sheaves on a point and Kazhdan-Lusztig bases

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March 21, 2021

Kazhdan-Lusztig/Chris-Ginzburg isomorphism

Let G be a connected, reductive, complex algebraic group with $[G, G]$ simply connected.

- $T \subseteq B = TU \subseteq G$, the Lie algebras are $\mathfrak{t} \subseteq \mathfrak{b} = \mathfrak{t} + \mathfrak{u} \subseteq \mathfrak{g}$.
- $X = X(T)$, $A = \mathbb{Z}[X(\mathbb{C}^*)] = \mathbb{Z}[v, v^{-1}]$.
- $W = W(G, T)$ is the Weyl group.
- $\mathcal{H}_W$ is the Iwahori-Hecke of W over A.
- $\mathcal{H} = A[X] \otimes_A \mathcal{H}_W$ is the extended, affine Hecke algebra.
- $\overline{G} = G \times \mathbb{C}^*$, $\overline{T} = T \times \mathbb{C}^*$. Note that $A[X] = \mathbb{Z}[X(\overline{T})]$.

- The Steinberg variety of G is

$$\begin{aligned} Z &= T^*(G/B) \times_{\mathfrak{g}} T^*(G/B) \\ &= \{\, (x, gB, hB) \in \mathfrak{g} \times G/B \times G/B \mid g^{-1}x, h^{-1}x \in \mathfrak{u} \,\}. \end{aligned}$$

- $z \in \mathbb{C}^*$ acts on $\mathfrak{g}$ via multiplication by z^{-2} and trivially on G/B.
- $\overline{G} = G \times \mathbb{C}^*$ acts diagonally on $\mathfrak{g} \times G/B \times G/B$.
- $K_{\overline{G}}(Z)$ is the Grothendieck "group" of the category of $\overline{G}$-equivariant coherent sheaves of $\mathcal{O}_Z$-modules – an A-algebra with convolution multiplication (Chriss-Ginzburg).

Theorem (Kazhdan-Lusztig, Chris-Ginzburg, Lusztig)

There is a isomorphism of A-algebras $\varphi \colon \mathcal{H} \xrightarrow{\cong} K_{\overline{G}}(Z)$

Extended affine Weyl group, Hecke algebra

- $W_{\mathrm{ex}} = X \rtimes W = \{ wt_\lambda \mid w \in W, \lambda \in X \}$ is the extended affine Weyl group; $wt_\lambda = t_{w(\lambda)} w$.
  - Length function ℓ, Bruhat order $\leq$, w_0, Kazhdan-Lusztig cells
  - $\Gamma = \{ z = xw_0y \mid \ell(z) = \ell(x) + \ell(w_0) + \ell(y) \}$ is the lowest two-sided cell.
- $\mathcal{H} = A[X] \otimes_A \mathcal{H}_W$ has bases:
  - Standard basis $\{ T_z \mid z \in W_{\mathrm{ex}} \}$
  - Bernstein basis $\{ \theta_\lambda T_w \mid \lambda \in X, w \in W \}$, $T_w \theta_\lambda = \theta_{w(\lambda)} T_w + *$
  - Kazhdan-Lusztig bases $\{ C_z \mid z \in W_{\mathrm{ex}} \}$, $\{ C'_z \mid z \in W_{\mathrm{ex}} \}$

Proposition (Lusztig, . . .)

The cell "ideal" corresponding to Γ the two-sided ideal $\mathcal{H} C'_{w_0} \mathcal{H}$.

$\mathcal{H}C_{w_0}\mathcal{H}$ and Schubert calculus?

- $Z_0 = \{0\} \times G/B \times G/B \subseteq Z$
- $\rho \in X$ is the sum of the fundamental dominant weights.
- $u\colon A[X] \otimes_{A[X]^W} A[X] \to \mathcal{H}C'_{w_0}\mathcal{H}$, $u(\theta_\lambda \otimes \theta_\mu) = \theta_{\lambda+\rho}C'_{w_0}\theta_{\mu+\rho}$

Proposition (Lusztig)

The mapping u is an A-algebra isomorphism and the diagram of A-algebra homomorphisms commutes:

$$\begin{array}{ccccc} A[X]\otimes_{A[X]^W} A[X] & \xrightarrow[\cong]{u} & \mathcal{H}C'_{w_0}\mathcal{H} & \hookrightarrow & \mathcal{H} \\ \cong\downarrow & & \cong\downarrow & & \cong\downarrow\varphi \\ K_{\overline{G}}(G/B\times G/B) & \xrightarrow[\cong]{} & K_{\overline{G}}(Z_0) & \hookrightarrow & K_{\overline{G}}(Z). \end{array}$$

Generalized Steinberg varieties

- $P = L_P U_P$, $Q = L_Q U_Q$ parabolic subgroups with $B \subseteq P \cap Q$.
- $\mathfrak{N} \subseteq \mathfrak{g}$, $\mathfrak{N}_P \subseteq \mathfrak{p}$, $\mathfrak{N}_Q \subseteq \mathfrak{q}$ are the respective nilpotent cones.
- $X^{PQ} = \{\, (x, gP, hQ) \mid g^{-1}x \in \mathfrak{N}_P, h^{-1}x \in \mathfrak{N}_Q \,\}$.
- $Y^{PQ} = \{\, (x, gP, hQ) \mid g^{-1}x \in \mathfrak{u}_P, h^{-1}x \in \mathfrak{u}_Q \,\}$.
- $Z^{PQ} = \{\, (x, gB, hB) \mid g^{-1}x \in \mathfrak{u}_P, h^{-1}x \in \mathfrak{u}_Q \,\}$.
- $X^{BB} = Y^{BB} = Z^{BB} = Z$ and there is a cartesian diagram:

$$\begin{array}{ccc} Z^{PQ} & \hookrightarrow & Z \\ \downarrow & & \downarrow \eta \\ Y^{PQ} & \hookrightarrow & X^{PQ}. \end{array}$$

Weyl groups W_P, W_Q; longest elements w_P, w_Q

Theorem

There is a commutative diagram of A-algebra homomorphisms:

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow[\cong]{\varphi} & K_{\overline{G}}(Z) \\ \downarrow \chi & & \downarrow \eta_* \\ C_{w_P}\mathcal{H}C_{w_Q} & \xrightarrow[\cong]{} & K_{\overline{G}}(X^{PQ}). \end{array}$$

Corollary

$C_{w_P}\mathcal{H}C_{w_Q} \cong K_{\overline{G}}(X^{PQ})$ *has standard, Bernstein, and* **Kazhdan-Lusztig bases**.

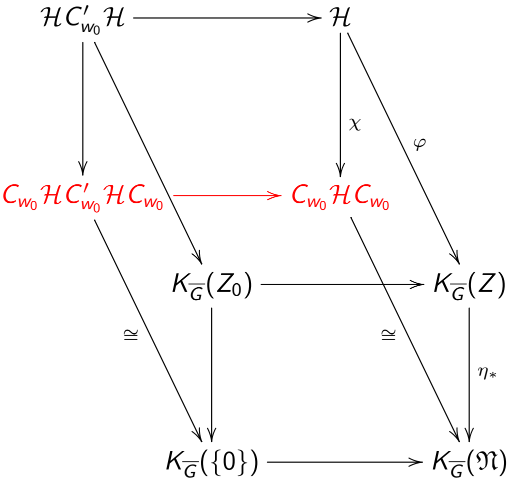
The case $P = Q = G$

From now on take $P = Q = G$.

- $X^{GG} = \{\,(x, G, G) \mid x \in \mathfrak{N}\,\} = \mathfrak{N}$.
- $Y^{GG} = \{\,(0, G, G)\,\} = \{0\}$.
- $Z^{GG} = \{\,(0, gB, hB)\,\} = Z_0$.
- There are commutative diagrams:

$$\begin{array}{ccccccccc} Z_0 & \hookrightarrow & Z & \overset{K_{\overline{G}}}{\rightsquigarrow} & K_{\overline{G}}(Z_0) & \hookrightarrow & K_{\overline{G}}(Z) & \overset{\mathcal{H}}{\longleftarrow} & \cdots \\ \downarrow & & \downarrow \eta & & \downarrow & & \downarrow \eta_* & & \\ \{0\} & \hookrightarrow & \mathfrak{N} & & K_{\overline{G}}(\{0\}) & \hookrightarrow & K_{\overline{G}}(\mathfrak{N}) & & \end{array}$$

The big picture



The inclusion $C_{w_0}\mathcal{H}C'_{w_0}\mathcal{H}C_{w_0} \subseteq C_{w_0}\mathcal{H}C_{w_0}$

- $K_{\overline{G}}(\{0\})$ is the representation ring of $\overline{G}$, so $C_{w_0}\mathcal{H}C'_{w_0}\mathcal{H}C_{w_0} \cong A[X]^W \cong Z(\mathcal{H})$.
- Define $D = C_{w_0}\theta_\rho C'_{w_0}\theta_\rho C_{w_0} \in C_{w_0}\mathcal{H}C'_{w_0}\mathcal{H}C_{w_0}$.
- X^+ is the set of dominant weights.
- $s_\mu \in A[X]^W$ is the "Schur polynomial."

Proposition

The rule $s_\mu \mapsto (-v)^{\ell(w_0)} s_\mu D$ $(\mu \in X^+)$ defines an A-linear isomorphism $A[X]^W \xrightarrow{\cong} C_{w_0}\mathcal{H}C'_{w_0}\mathcal{H}C_{w_0}$ that is compatible with the various commutative diagrams above.

- For $\mu \in X^+$, m_μ is the element with minimal length in $Wt_\mu W$.
- (Ostrik) $C_{w_0} \mathcal{H} C_{w_0} \cong K_{\overline{G}}(\mathfrak{N})$ has
  - standard basis $\{ T_\mu = C_{w_0} T_{m_\mu} C_{w_0} \mid \mu \in X^+ \}$,
  - Kazhdan-Lusztig basis $\{ C'_\mu = C_{w_0} C'_{m_\mu} C_{w_0} \mid \mu \in X^+ \}$

Theorem*

Suppose $\mu \in X^+$. Then

$$(-1)^{\ell(w_0)} s_\mu D = C'_{\mu+2\rho} \in C_{w_0} \mathcal{H} C_{w_0}.$$

Thus, $\{ C'_{\mu+2\rho} \mid \mu \in X^+ \}$ is an A-basis of $C_{w_0} \mathcal{H} C'_{w_0} \mathcal{H} C_{w_0}$ (and so determines an A-basis of $K_{\overline{G}}(Y^{GG})$).

What about $K_{\overline{G}}(Y^{PQ})$? $K_{\overline{G}}(Z^{PQ})$? ...

Thank you!