Problem 1.

(a) In this representative agent model, the government sets its spending on public good $g_t$ so that it maximizes the consumer’s life-time utility. Consider the following social planner’s problem:

\[
\max_{\{c_t, l_t, g_t\}} \beta^t [u(c_t) + v(l_t) + \phi(g_t)]
\]

s.t. $c_t + g_t \leq z_t \forall t$ \hspace{1cm} (2)

\[n_t + l_t = 1 \forall t\] \hspace{1cm} (3)

Since the sum of strictly increasing utility functions is also a strictly increasing utility function, the resource constraint (2) will hold with equality. So we can use the constraints to substitute for $c_t$ in the utility function:

\[
\max_{\{c_t, l_t, g_t\}} \beta^t [u(z_t(1 - l_t) - g_t) + v(l_t) + \phi(g_t)]
\]

The FOCs for this problem are:

\[l_t : -z_t u'(z_t(1 - l_t) - g_t) + v'(l_t) = 0 \quad \text{for } \forall t\] \hspace{1cm} (4)

\[g_t : -u'(z_t(1 - l_t) - g_t) + \phi'(g_t) = 0 \quad \text{for } \forall t\] \hspace{1cm} (5)

Given the actual functional form of utility functions we can use conditions (4) and (5) to solve for optimal values of leisure, public good consumption and consumption good consumption. Note that for $\forall t$ we will have

\[l_t^* = l(z_t)\] \hspace{1cm} (6)

\[g_t^* = g(z_t)\] \hspace{1cm} (7)

\[c_t^* = c(z_t)\] \hspace{1cm} (8)

\[n_t^* = 1 - l_t^* = n(z_t)\] \hspace{1cm} (9)

To find the real wage and the real interest rate we have to solve the competitive equilibrium problem. Note that the consumer takes the government spending/public good consumption as given.

The consumer’s problem is:

\[
\max_{\{c_t, l_t, s_{t+1}\}} \beta^t [u(c_t) + v(l_t) + \phi(g_t)]
\]

s.t. $c_t + s_{t+1} \leq w_t(1 - l_t) + (1 + r_t)s_t - \tau_t \quad \text{for } \forall t$ \hspace{1cm} (11)

The Lagrangian for this problem is:
\[ L = \sum_{t=0}^{\infty} \beta^t [u(c_t) + v(l_t) + \phi(g_t)] + \sum_{t=0}^{\infty} \lambda_t(w_t(1-l_t) + (1+r_t)s_t - \tau_t - c_t - s_{t+1}) \]

The FOCs for the consumer’s problem for every \( t \) are:

\[ \frac{v(l_t)}{u(c_t)} = w_t \]  
(12)

\[ \frac{\beta u(c_{t+1})}{u(c_t)} = \frac{1}{1 + r_{t+1}} \]  
(13)

The firm’s problem for every is:

\[ \max_{n_t} z_t n_t - w_t n_t \]  
(14)

Labor supply is infinitely elastic at following wage:

\[ w_t = z_t \]

The government’s budget constraint is balanced

\[ g_t + (1 + r_t)l_t = \tau_t + b_{t+1} \]  
(15)

and the markets for labor and bonds clear every period \( t \):

\[ n_t + l_t = s_{t+1} \]

In equilibrium, we will have that

\[ \frac{v(l^*_t)}{u(c^*_t)} = z_t \]

\[ \frac{w(c^*_t)}{\beta u(c^*_{t+1})} - 1 = r^*_{t+1} \]

In the future, you don’t have to solve for the C.E. first-order conditions, you can just use (12)-(13).

From the social planner’s problem we know that (6)-(9) will depend on the values taken by exogenous productivity shock \( \{z_t\}_{t=0}^{\infty} = \{z^*, z^{**}, z^*, z^{**}, \ldots\} \), where \( z^* > z^{**} \). Thus, the real wage and the real interest rate will also be a function of \( z_t \). As productivity shock follows two cycles, consumption, leisure, employment, government spending and the real interest rate will do the same. However, so far we cannot determine whether these values will move procyclically or countercyclically.

(b)

To determine the effect of total productivity differential change on the equilibrium allocation and prices, I totally differentiated (4) and (5) and got two unknowns and two
equations. I will suppress the time subscript as this will not be relevant for my analysis because the FOCs hold for every period $t$.

$$-u'dz - z(1-l)u''dz + z^2u''dl + zu''dg + v''dl = 0$$
$$-(1-l)u''dz + zu''dl + u''dg + \phi''dg = 0$$

Thus, $A \mathbf{x} = \mathbf{b}$:

$$(1-l)u''dz + zu''dl + u''dg + \phi''dg = 0$$

We have that $\det A = v''u'' + z^2u''\phi'' + v''\phi'' > 0$

due to strict concavity of $u(\cdot), v(\cdot), \phi(\cdot)$.

Then the impact on leisure depends on income (leisure is a normal good) and substitution (as the real wage increases, leisure becomes relatively more expensive) effects, thus, the total effect is ambiguous:

$$\frac{dl}{dz} = \frac{z(1-l)\phi''u'' + u'(u'' + \phi'')}{\det A}$$

The effect of total productivity shock on employment is also ambiguous:

$$\frac{dn}{dz} = \frac{-dl}{dz} = -\frac{z(1-l)\phi''u'' + u'(u'' + \phi'')}{\det A}$$

Government spending increases with an increase in total factor productivity:

$$\frac{dg}{dz} = \frac{(1-l)v'' - zu'}{\det A} > 0$$

Also consumption increases as total productivity increases due to income effect (consumption is a normal good) and substitution effect (leisure is more expensive now $\frac{dc}{dz} = 1 > 0$).

$$\frac{dc}{dz} = \frac{d}{dz}(z(1 - l) - g) = (1 - l) + \frac{dl}{dz} - \frac{dg}{dl} =$$
$$= \frac{v''\phi''(1-l) - zu'\phi''}{\det A} > 0$$

The output will increase as a result of an increase in total factor productivity:

$$\frac{dy}{dz} = \frac{d}{dz}(z(1 - l)) = (1 - l) - z \frac{dl}{dz} =$$
$$= \frac{(1-l)(v''u'' + v''\phi'') - zu'(u'' + \phi'')}{\det A} > 0$$

One way to think of these results is that there are two goods in this economy, private consumption goods and public goods. When there is an increase in $z$, consumption of both goods ($c$ and $g$) rises, just as private consumption rises when it is the only good. Just as in the case where $g$ is exogenous, there are income and substitution effects on the quantity of
leisure consumed, and leisure and employment may rise or fall.

So now there are basically two periods in this economy: even periods with high total factor productivity $z^*$ and odd periods with low total factor productivity $z^{**}$. Thus,

$$ l_t, n_t = \text{undetermined for any } \forall t $$

$$ c_t \implies t \text{ is even: high consumption } c^* $$

$$ t \text{ is odd then low consumption } c^{**} $$

$$ \{c_t\}_{t=0}^\infty = \{c^*, c^{**}, c^*, c^{**}, \ldots\} $$

$$ g_t \implies t \text{ is even: high government spending } g^* $$

$$ t \text{ is odd then low government spending } g^{**} $$

$$ \{g_t\}_{t=0}^\infty = \{g^*, g^{**}, g^*, g^{**}, \ldots\} $$

$$ y_t \implies t \text{ is even: high consumption } y^* $$

$$ t \text{ is odd then low consumption } y^{**} $$

$$ \{y_t\}_{t=0}^\infty = \{y^*, y^{**}, y^*, y^{**}, \ldots\} $$

As for the interest rate:

$t$ is even:

$$ r^*_t + 1 = \frac{u(c^*)}{u(c^*_t)} - 1 = \frac{u(c^*)}{u(c^{**})} - 1 = r_1 $$

$t$ is odd:

$$ r^{**}_t + 1 = \frac{u(c^{**})}{u(c^{**}_t)} - 1 = \frac{u(c^{**})}{u(c^*)} - 1 = r_2 $$

Thus, $r_1 < r_2$ because of the diminishing marginal utility property (i.e., the strict concavity of the utility function).

$$ \{r_t\}_{t=0}^\infty = \{r_1, r_2, r_1, r_2, \ldots\} $$

To summarize: in even periods when the economy is hit by positive shock $z^*$, consumption, government spending and output will increase. However, the effect on leisure is ambiguous due to the opposite income and substitution effect of the resulted increase in the real wage. The intertemporal decision in this model is concerned with bond market: in even periods consumption is too high and the consumer wants to smooth his consumption by saving more, i.e., he buys bonds, the demand for bonds increases, the price of bonds increases, and thus the real interest rate falls. In other words, the real interest rate has to decrease in order to dissuade the consumer from saving too much. In odd periods, everything is reversed.

(c) It does not matter for the C.E. whether the government chooses tax scheme (i) or (ii). In this model the Ricardian Equivalence hold and thus timing and amount of taxation do not matter. The timing and amount of taxes matters only for saving and borrowing decisions, but not for employment, leisure, consumption, output, the real wage and the real interest rate.

The government deficit:

$$ \text{GovDef}_t = g_t + (1 + r_t) b_t - \tau_t $$
The government debt:

\[ \text{GovDebt}_t = b_{t+1} \]

(i) \( \tau_t = g_t, \forall t \)

In this case the government deficit is equal to:

\[ \text{GovDef}_t = (1 + r_t)b_t = \text{GovDebt}_t = b_{t+1} \]

from the government budget constraint.

Note that if \( b_0 = 0 \) then \( b_t = 0 \) for \( \forall t \), the government balances its budget:

\[ b_{t+1} = (1 + r_t)b_t = 0 \text{ for } \forall t, \]

and government budget deficit and government debt is zero each period \( t \).

(ii) \( \tau_t = \tau^*, \forall t \)

Then the government deficit is:

\[ \text{GovDef}_t = g_t + (1 + r_t)b_t - \tau^* \]

and the government debt is:

\[ \text{GovDebt}_t = b_{t+1}, \]

and thus, they are equal:

\[ g_t + (1 + r_t)b_t - \tau^* = b_{t+1} \]

because the government has to balance its budget every period.

To determine the path followed by the government deficit and the government debt, we have to solve the first-order difference equation:

\[ b_{t+1} = (1 + r_t)b_t + g_t - \tau^* \]

If \( b_0 = 0 \) and

\[ a_1 = g^* - \tau^* \]

and \( a_2 = g^{**} - \tau^* \) then

\[ b_1 = a_1 \]

\[ b_2 = (1 + r_2)a_1 + a_2 \]

\[ b_3 = a_1(1 + r_1)^5(1 + r_2)^5 + a_2(1 + r_1) \]

\[ b_4 = a_1(1 + r_2)^5(1 + r_1)^5 + a_2(1 + r_1)^5(1 + r_2)^5 \]

\[ b_5 = a_1(1 + r_1)^5(1 + r_2)^5 + a_2(1 + r_1)(1 + r_1)^5(1 + r_2)^5 \]

Then there are two cases:

(1) when \( t \) is odd (then \( t + 1 \) is even):

\[ b_{t+1} = a_1(1 + r_2) \left[ \frac{(1 + r_1)^5(1 + r_2)^5}{s=0} \right] + a_2(1 + r_1)^5(1 + r_2)^5 = \]

\[ = (a_1(1 + r_2) + a_2) \left[ \frac{(1 + r_1)(1 + r_2)}{s=0} \right] - 1 \]

\[ = \left( [g^* - \tau^*(1 + r_2)](1 + r_2) + (g^{**} - \tau^*) \right) \left[ \frac{(1 + r_1)(1 + r_2)}{s=0} \right] - 1 \]

(1) when \( t \) is even (then \( t + 1 \) is odd):

...
Explanation: In part (i) the government sets taxes to be equal exactly to government spending, thus there is no need for bonds to be issued. However, in part (ii) the situation is different. If $g^* > \tau^* > g^{**}$ the government will either issue bonds or buy bonds from the consumer, but this depends on the path of the government deficit/debt $b_{t+1}$. 

\[
b_{t+1} = \sum_{s=0}^{\infty} \frac{X^s}{(1 + r_1)^s(1 + r_2)^s + a_2(1 + r_1)} \frac{X^{-1}}{(1 + r_1)^s(1 + r_2)^s} = \\
= (1 + r_1) \left[ a_1(1 + r_2) + a_2 \right] \frac{[(1 + r_1)(1 + r_2)]^{\frac{1}{2}-1} - 1}{(1 + r_1)(1 + r_2) - 1} = \\
= (1 + r_1) \left[ (g^* - \tau^*)(1 + r_2) + (g^{**} - \tau^*) \right] \frac{[(1 + r_1)(1 + r_2)]^{\frac{1}{2}-1} - 1}{(1 + r_1)(1 + r_2) - 1}
\]