(1) **Optimal Growth with a proportional wealth tax.**

Consider the following Economy:

**Time:** Discrete infinite horizon

**Demography:** Continuum of mass 1 of (representative) consumer/worker households, a Government which has to meet a fixed per household expenditure $g$ and a large number of profit maximizing firms

**Preferences:** the instantaneous household utility function over, consumption, $c$ is $u(.)$. Which is strictly increasing and strictly concave. The discount factor is $\beta \in (0, 1)$.

**Technology:** There is a constant returns to scale technology over capital labor such that per capita output $f(k)$ where $k$ is capital input per unit of labor.

**Endowments:** Households’ initial capital stock is $k_0$, each household has 1 unit of time which it can devote in any proportions to labor or leisure. Assume $f(k_0) + (1 - \delta) k_0 > g$ (Required to get the economy off the ground.)

(a) Derive for the system of equations that specify the dynamics of the Planner’s solution for this economy.

(b) Now consider the decentralized economy. If the government’s only means of collecting revenue is a proportional tax, $\tau_t$, on wealth (i.e. capital holdings) and the budget is balanced every period, write down the households’ and firms’ optimization problems.

(c) Define a competitive equilibrium and solve for the system of equations that specify equilibrium dynamics.

(d) what restriction on the second derivative of $f$ and on $g$ is required for the existence of a saddlepath stable steady state?

(e) Show diagrammatically how an increase in $g$ affects the saddlepath stable steady-state. Explain
(f) Explain why the response to a change in $g$ in the equilibrium allocation is not the same as that in the Planner’s allocation.

**Sketch of Solution to Q1:**

(a) Planner solves

$$\max_{(k_t)} \sum_{0}^{\infty} \beta^t u(f(k_t) - k_{t+1} + (1 - \delta)k_t - g)$$

Leading to:

$$\frac{u'(c_t)}{u'(c_{t-1})} = \frac{1}{\beta[f'(k_t) + 1 - \delta]}$$

$$c_t = f(k_t) - k_{t+1} - (1 - \delta)k_t - g$$

(b) H/H problem:

$$\max_{(k_t)} \sum_{0}^{\infty} \beta^t u(w_t + r_t k_t - k_{t+1} + (1 - \delta - \tau_t)k_t)$$

Leading to:

$$\frac{u'(c_t)}{u'(c_{t-1})} = \frac{1}{\beta[r_t + 1 - \delta - \tau_t]}$$

$$c_t = w_t + r_t k_t - k_{t+1} + (1 - \delta - \tau_t)k_t$$

Firms problem implies $r_t = f'(k_t^f)$, $w_t = f(k_t^f) - r_t f'(k_t^f)$.

(c) A Balanced budget Competitive equilibrium is an allocation and sequence of prices $\{w_t, r_t\}$ such that given prices the allocation solves the firms and workers problems and markets clear.

Market clearing: $k_t = k_t^f = \bar{k}_t$ Balanced budget requires, $\tau_t \bar{k}_t = g$, where $\bar{k}_t$ is the average capital holdings.

System is

$$\frac{u'(c_t)}{u'(c_{t-1})} = \frac{1}{\beta[f'(k_t) + 1 - \delta - g/k_t]}$$

$$c_t = f(k_t) - k_{t+1} - (1 - \delta)k_t - g$$
(d) For existence and uniqueness of saddlepath stable steady-state we need
\[ \frac{d}{dk} [f'(k) + 1 - \delta - g/k] = f''(k) + g/k^2 < 0 \]

(e) The vertical line should shift left. The curve will shift down as it does in the planner’s model.

(f) The tax is distortionary, households do not take account of the benefit to others of reduces taxes by increasing their own capital holdings so there is too little capital in equilibrium.

(2) Consider the following model of search with repeated promotion.

**Time:** Discrete, infinite horizon.

**Demography:** Single worker who lives for ever.

**Preferences:** The worker is risk-neutral (i.e. \( u(x) = x \)). He discounts the future at the rate \( r \).

**Endowments:** When unemployed: The worker receives income \( b \) per period when unemployed. Also, with probability \( \alpha \) he gets to sample a wage from the continuous distribution \( F \) with support \((b, \tilde{w}] \) where \( \tilde{w} > b \). Define \( \bar{w} \equiv \int_b^{\tilde{w}} wdF(w) \), the unconditional mean of the distribution.

When employed: The worker receives his current wage and is subject to two possible changes in his status. With probability \( \lambda \) he is laid-off (returns to unemployment). With probability \( \gamma \) he gets promoted. (So that with probability \( 1 - \lambda - \gamma \) there is no change in status.) Promotion means that the wage is increased by the factor \( \phi > 1 \). Promotion can happen any number of times but, of course, once laid-off a worker has to start all over again on the employment ladder. Assume that \( r + \lambda > \gamma(\phi - 1) \) (It keeps the value functions bounded.)

(a) Write down the relevant value functions (Hint: you only need 2)

(b) Solve for the value to employment at any given wage as a function of the wage and the value to unemployment. (Hint, the value function is linear in the wage and so you can use the method of undetermined coefficients).

(c) Derive for the reservation wage equation.
(d) Derive an equation in terms of the model parameters and $\hat{w}$ for the critical value, $\gamma^*$, of $\gamma$ such that the reservation wage, $w^* = b$. Explain.

**Sketch of Solution to Q2.**

\[ rU = b + \alpha \mathbb{E}_w \max \{V(w) - U, 0\} \quad (1) \]

\[ rV(w) = w + \lambda (U - V(w)) + \gamma (V(\phi w) - V(w)) \]

(b)

\[ (r + \lambda + \gamma) V(w) = w + \lambda U + \gamma V(\phi w) \]

Assume $V(w) = A + Bw$ and compare coefficients (there might be a more straightforward way):

on $w^0$,

\[ A(r + \lambda + \gamma) = \lambda U + \gamma A \]

on $w$,

\[ B(r + \lambda + \gamma) = 1 + \gamma \phi B \]

so

\[ A = \frac{\lambda U}{r + \lambda}, \quad B = \frac{1}{r + \lambda + \gamma(1 - \phi)} \]

(c) From (1):

\[ rU = b + \alpha \int_{w^*}^{\hat{w}} V(w) - UdF(w) \quad (2) \]

where $w^*$ is such that

\[ V(w^*) = U = \frac{\lambda U}{r + \lambda} + \frac{w^*}{r + \lambda + \gamma(1 - \phi)} \]

so

\[ rU = \frac{(r + \lambda) w^*}{r + \lambda + \gamma(1 - \phi)} \]

Subbing in to (2) we get

\[ \frac{(r + \lambda) w^*}{r + \lambda + \gamma(1 - \phi)} = b + \frac{\alpha}{r + \lambda + \gamma(1 - \phi)} \int_{w^*}^{\hat{w}} w - w^* dF(w) \]

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or

\[ w^* = \frac{[r + \lambda + \gamma(1 - \phi)] b}{(r + \lambda)} + \frac{\alpha}{r + \lambda} \int_{w^*}^{\bar{w}} w - w^* dF(w) \]

(d) setting \( w^* = b \)

\[ (r + \lambda) b = [r + \lambda + \gamma(1 - \phi)] b + \alpha(\hat{w} - b) \]

implies

\[ \gamma^* = \frac{\alpha(\hat{w} - b)}{b(\phi - 1)} \]