Macroeconomics I: Mid-Term Exam
Diamond overlapping generations model with special storage technology.

**Time:** discrete, infinite horizon

**Demography:** A mass $N_t \equiv N_0 (1 + n)^t$ of newborns enter in every period. Everyone lives for 2 periods except for the first generation of old people.

**Preferences:** for the generations born in and after period 1;

$$U_t(c_{1,t},c_{2,t+1}) = u(c_{1,t}) + \beta u(c_{2,t+1})$$

where $c_{i,t}$ is consumption in period $t$ and stage $i$ of life. For the initial old generation $\bar{U} (c_{2,1}) = u(c_{2,1})$, the function $u(.)$ is twice continuously differentiable, increasing and strictly concave.

**Productive technology:** Firms have access to the technology $F(K,L)$ where $K$ is the capital stock and $L$ is labor. (It will be convenient to write period $t$ output per worker as $f(k_t)$ where $k_t$ is per worker capital stock at the firm.)

**Special Storage technology:** This technology is available to the households. Rather than save in capital, households can bury consumption goods at the end the first period of life and dig up $\gamma$ units of the consumption good in old age for every unit buried. (What makes it special is that $\gamma$ can be larger than $1 + n$.) NB: An individual household can use both ways of saving simultaneously. They can create capital out of some consumption goods and can bury others.

**Endowments:** Everyone has one unit of labor services when young. (Old people cannot work so they have to rely on earnings from renting capital and/or from using the storage technology.) The first generation of old have $(1 + n)k_1$ units of capital each.

**Institutions:** There are competitive markets, for labor, physical capital and consumption goods. Using the consumption good as the numeraire, let the per unit wage in period $t$ be $w_t$ and the gross return on capital rented in period $t$ be $R_t$.

1. Write out and solve the problems faced by generation $t$ individuals and firms in this economy. (ignore inside money).

$$\max_{c_{1,t},c_{2,t+1},s_{t+1},q_{t+1}} u(c_{1,t}) + \beta u(c_{2,t+1})$$

$$c_{1,t} = w_t - s_{t+1} - q_{t+1}$$

$$c_{2,t+1} = R_{t+1}s_{t+1} + \gamma q_{t+1}$$

Where $s_{t+1}$ is saving in capital and $q_{t+1}$ is saving using the storage technology. After substituting out $c_{1,t}$ and $c_{2,t+1}$ the FOCs become

$$s_{t+1} : \quad -u'(c_{1,t}) + \beta R_{t+1}u'(c_{2,t+1}) = 0$$

$$q_{t+1} : \quad -u'(c_{1,t}) + \beta \gamma u'(c_{2,t+1}) = 0$$

**Firms:**

$$R_t = f'(k_t)$$

$$w_t = f(k_t) - k_t f'(k_t)$$
2. Write down the market clearing conditions and define a competitive equilibrium.

Market Clearing:

- Capital: 
  \[(1 + n)k_{t+1} = s_{t+1}\]
- Goods: 
  \[f(k_t) + \frac{\gamma q_t}{1 + n} = c_{1t} + \frac{c_{2t}}{1 + n} + q_{t+1} + s_{t+1}\]
- (Labor: \[N_t = 1\])

(In the representative household version the labor market clearing is really unnecessary.)

**Definition:** A perfect foresight competitive equilibrium is an allocation, \(\{c_{1t}, c_{2t}, q_t, k_t\}\) and prices \(\{R_t\}\) such that given prices, the allocation solves the households’ and firms’ problems and, markets clear.

3. What condition has to hold on \(R_t\) and \(\gamma\) for both technologies to be used?

For both technologies to be used we need \(R_t = \gamma\).

4. Which equations (there will be 3) characterize an equilibrium in which both technologies are used? What do the dynamics of such an equilibrium look like?

A characterization of an equilibrium in which both technologies are being used (in 3 equations) is

\[
\begin{align*}
  f'(k_t) &= \gamma \\
  f'(k_{t+1}) &= \gamma \\
  u'(f(k_t) - k_t f'(k_t) - (1 + n)k_{t+1} - q_{t+1}) &= \beta \gamma u'(f'(k_{t+1})(1 + n)k_{t+1} + \gamma q_{t+1})
\end{align*}
\]

The equilibrium has no dynamics, \(k_t\) and \(q_t\) are both constant through time.

5. Write down the problem faced by a Social Planner who weights all generations equally.

The Planner’s Lagrangian is

\[
\mathcal{L} = u(c_{21}) + \sum_{t=1}^{\infty} u(c_{1t}) + \beta u(c_{2t+1}) + \lambda_t \left[ f(k_t) + \frac{\gamma q_t}{1 + n} - c_{1t} - \frac{c_{2t}}{1 + n} - q_{t+1} - (1 + n)k_{t+1} \right]
\]

6. Obtain first-order conditions on the basis that \(\gamma > 1 + n\). Will the Planner use the storage technology?

FOCs:

- \(c_{21}\): 
  \[u'(c_{21}) - \frac{\lambda_1}{1 + n} = 0\]
- \(c_{1t}\): 
  \[u'(c_{1t}) - \lambda_t = 0\]
- \(c_{2t}\): 
  \[\beta u'(c_{2t}) - \frac{\lambda_t}{1 + n} = 0\]
- \(k_t\): 
  \[\lambda_t f'(k_t) - \lambda_{t-1}(1 + n) = 0\]
- \(q_t\): 
  \[\frac{\lambda_t \gamma}{1 + n} - \lambda_{t-1} = 0\]

If \(\gamma > 1 + n\), the Planner will use both technologies and pick \(k_t\) to solve \(f'(k_t) = \gamma\). The Planner’s economy will grow without bound but the market economy will not. In the Planner’s economy, \(k_t\) will be constant through time but \(q_t\) will grow.

Now assume that \(u(c) = \ln(c)\) (natural logarithm) and \(f(k) = zk^\alpha\) where \(a < 1\).
7. Obtain formulae for $c_{1,t}$ and $c_{2,t+1}$ in an equilibrium in which both technologies are being used. What is the relationship between consumption and $\gamma$?

Given these preferences we get

$$c_{1,t} = \frac{w_t}{1+\beta}$$

$$c_{2,t} = \frac{\beta w_t}{1+\beta}$$

Now,

$$w_t = z(1-\alpha)k_t^\alpha$$

and as $f'(k_t) = \gamma$,

$$k_t = \left(\frac{\alpha z}{\gamma}\right)^\frac{1}{1-\alpha}$$

so

$$w_t = z(1-\alpha)\left(\frac{\alpha z}{\gamma}\right)^\frac{\alpha}{1-\alpha}$$

this means that consumption in both periods is decreasing in $\gamma$. Which seems odd but a high value of $\gamma$ means the workers save manly in the storage technology which means there is little capital for the next generation to work with and wages fall and that is the only source of income to the young. The Planner can reallocate income between the generations.

8. If $\gamma > 1 + n$ is the equilibrium Pareto optimal?

This is actually kind of tricky. The first generation of old just use $k_0$ to create jobs. After that we want to use the storage technology to create consumption for everyone so that the young will save enough in the future to sustain the economy. We need to give extra consumption to the young which means some old generation will have to give up consumption relative to the laissez-faire market economy. This implies that the market equilibrium is P.O.