Macroeconomics I: Mid-Term Exam

Diamond Overlapping generations model with exogenous growth

**Time:** discrete, infinite horizon

**Demography:** A mass $N$ of newborns enter in every period (i.e. no population growth). Everyone lives for 2 periods except for the first generation of old people.

**Preferences:** for the generations born in and after period 1;

$$U_t(c_{1,t}, c_{2,t+1}) = \ln(c_{1,t}) + \beta \ln(c_{2,t+1})$$

where $c_{i,t}$ is consumption in period $t$ and stage $i$ of life. For the initial old generation $U(2,1) = \ln(c_{2,1})$.

**Productive technology:** Firms have access to the technology $z_t F(K_t, L_t) = z_t K_t^\alpha L_t^{1-\alpha}$ where $K_t$ is the period $t$ capital stock, $L_t$ is period $t$ labor, $\alpha \in (0, 1)$ and $z_t$ is the period $t$ total factor productivity (TFP). The gross growth rate of $z_t = \frac{z_{t+1}}{z_t} = 1 + \gamma$ so that $\gamma$ is the net growth rate of TFP. It will be convenient to use the period $t$ output per worker, $y_t = z_t k_t^\alpha$, where $k_t$ is per worker (i.e. per young person) capital stock at the firm.

**Endowments:** Everyone has one unit of labor services when young. (Old people cannot work so they have to rely on earnings from renting capital.) The first generation of old have $k_1$ units of capital each.

**Institutions:** There are competitive markets, for labor, physical capital and consumption goods. Using the consumption good as the numeraire, let the per unit wage in period $t$ be $w_t$ and the gross return on capital rented in period $t$ be $R_t$.

1. Write out and solve the problems faced by generation $t$ individuals and firms in this economy (ignore inside money).

Individual’s problem is:

$$\max_{s_{t+1}} \log(w_t - s_{t+1}) + \beta \log(R_{t+1}s_{t+1})$$

Leads to

$$s_{t+1} = \frac{\beta w_t}{1 + \beta}$$

Firm’s problem:

$$\max_{k_t} z_t k_t^\alpha - R_t k_t - w$$

So,

$$R_t = \alpha z_t k_t^{\alpha-1}$$

and

$$w_t = z_t k_t^\alpha - \alpha z_t k_t^{\alpha} = (1 - \alpha) z_t k_t^\alpha$$
2. Write down the market clearing condition for capital and define a competitive equilibrium.

Market clearing in capital implies:

\[ k_{t+1} = s_{t+1} = \frac{\beta z_t (1 - \alpha) k_t^\alpha}{1 + \beta}. \]

**Definition:** A competitive equilibrium is an allocation, \( \{k_t, c_1, c_2\} \) and prices, \( \{w_t, R_t\} \) such that given prices the allocation solve the individual’s and firm’s problems and markets clear.

As TFP grows at the rate \( \gamma \), there is no steady state. Instead, we will look for a “balanced growth path” (BGP). On the BGP all variables we are interested in, grow at a fixed rate. Let \( G_x \) be the notation for the gross growth rate of variable \( x = k, c_1, c_2 \) etc. So, for example \( G_k = \frac{k_{t+1}}{k_t} \) for all \( t \).

3. In terms of \( \gamma \), find \( G_k \) and show that \( G_k = G_y \).

From the market clearing condition

\[ G_k = \frac{k_{t+1}}{k_t} = \frac{\beta z_t (1 - \alpha) k_t^\alpha}{1 + \beta}, \]

\[ = \frac{z_{t+1} k_t^\alpha}{z_t k_{t-1}^\alpha} = G_y = (1 + \gamma) G_k^\alpha \]

Solving,

\[ G_k = G_y = (1 + \gamma)^{\frac{1}{1 - \alpha}}. \]

4. Show that \( G_R \), the growth rate of the gross interest rate (or rental rate of capital) equals 1 so that \( R_t = \bar{R} \) for all \( t \)?

\[ G_R = \frac{\alpha z_t k_t^{\alpha-1}}{\alpha z_{t-1} k_{t-1}^{\alpha-1}} = (1 + \gamma) G_k^{\alpha-1} = 1. \]

5. Solve for expressions for the per worker capital stock, \( k_t \) and \( \bar{R} \) as a function of \( z_t \) and model parameters.

\[ G_k = \frac{k_{t+1}}{k_t} = \frac{\beta z_t (1 - \alpha) k_t^\alpha}{(1 + \beta) k_t} \]

Which implies

\[ k_t = \left[ \frac{\beta z_t (1 - \alpha)}{G_k (1 + \beta)} \right]^{\frac{1}{1 - \alpha}} \]

\[ = \left[ \frac{\beta z_t (1 - \alpha)}{(1 + \gamma)^{\frac{1}{1 - \alpha}} (1 + \beta)} \right]^{\frac{1}{1 - \alpha}} \]

Now

\[ \bar{R} = \alpha z_t k_t^{\alpha-1} = \left( \frac{\alpha}{1 - \alpha} \right) \left( \frac{1 + \beta}{\beta} \right) (1 + \gamma)^{\frac{1}{1 - \alpha}} \]
6. Write down and solve the (per young person) problem faced by a Social Planner who weights all generations equally.

\[
\max_{\{c_{1t}, c_{2t}, k_{t+1}\}} \log(c_{21}) + \sum_{t=1}^{\infty} \{\log(c_{1t}) + \beta \log(c_{2t+1}) + \lambda_t [z_t k_t^\alpha - k_{t+1} - c_{1t} - c_{2t}]\}
\]

The first-order conditions are:

\[
\begin{align*}
\frac{1}{c_{21}} - \lambda_1 &= 0 \\
\frac{1}{c_{1t}} - \lambda_t &= 0 \\
\frac{\beta}{c_{2t}} - \lambda_t &= 0 \\
\lambda_t \alpha z_t k_t^{\alpha-1} - \lambda_{t-1} &= 0
\end{align*}
\]

Planner’s allocation satisfies

\[
\begin{align*}
\frac{c_{2t}}{c_{2t}} &= \beta c_{1t} \\
\alpha z_t k_t^{\alpha-1} &= \frac{\lambda_{t-1}}{\lambda_t} = \frac{c_{2t}}{c_{2t-1}} = G_{c_2} = G_y = (1 + \gamma)^{\frac{1}{1-\alpha}}
\end{align*}
\]

7. Given that on any BGP the growth rates of consumption, \(G_{c_1}\) and \(G_{c_2}\) are equal to \(G_y\), under what condition on parameters does the first welfare theorem hold?

Compare the last result to the value of \(\bar{R}\) for the market economy. Taking account of the impact of transfers on the initial old people means that whenever \(\bar{R} \geq (1 + \gamma)^{\frac{1}{1-\alpha}}\) the first welfare theorem holds. This requires that

\[
\left( \frac{\alpha}{1-\alpha} \right) \left( \frac{1 + \beta}{\beta} \right) \geq 1.
\]

When this condition does not hold there is over accumulation of capital and some generation of old people can get extra consumption.