(1) Optimal Growth with a consumption externality.

Consider the following Economy:

**Time:** Discrete; infinite horizon

**Demography:** Continuum of mass 1 of (representative) consumer/worker households, and a large number of profit maximizing firms, owned jointly by the households.

**Preferences:** the instantaneous household utility function over, individual consumption, \( c \), and average consumption \( \bar{c} \) is \( u(c) + v(\bar{c}) \). Where \( u(.) \) and \( v(.) \) are both strictly increasing, \( u(.) \) is strictly concave and \( v(.) \) is concave. The discount factor is \( \beta \in (0, 1) \). (Note: the sum of concave functions is concave.)

**Technology:** There is a constant returns to scale technology over capital and labor such that output per unit of labor employed is \( f(k) \), where \( k \) is capital input per unit of labor. Capital depreciates at the rate \( \delta < 1 \).

**Endowments:** Households’ initial capital stock is \( k_0 \), each household has 1 unit of labor.

(a) Write down the planner’s problem in recursive form and solve for the system of equations that governs the dynamics of His/Her solution.

(b) Now consider the decentralized economy. Write down the problem faced by the representative household, and the representative firm. Define and solve for a characterization of competitive equilibrium. (You can use either the sequence of markets or recursive competitive equilibrium.)

(c) Compare the saddlepath-stable steady-states under the two different institutional arrangements in (a) and (b) (i.e. planner and competitive equilibrium) and comment on your answer.

(d) If \( u(c) = \log(c) \) and \( v(c) = c \), comment on the difference in the out-of-steady-state dynamics. In particular, which saddle path should be steeper? Comment on your answer.
(2) Diamond Coconut Economy with Fog

**Time:** Discrete, infinite horizon

**Geography:** A single trading island and a large number of potential production islands. Each island has its own breed of tree that grow to an island specific height. So, all the coconut trees on any one island grow to the same height. The tree heights, \( c \), across islands are distributed \( F \). The support of \( F \) is \( (0, \tilde{c}) \).

**Demography:** A mass of 1 of ex ante identical individuals with infinite lives

**Preferences:** The common discount rate is \( r \), consumption of own produce yields 0 utils, consumption of anyone else’s output yields \( u > 0 \) utils.

**Productive Technology:** On any island individuals come across a tree with a coconut with probability \( \alpha \) each period. The cost of harvesting the coconut on an island with tree height \( c \), is \( c \) utils.

**Matching Technology:** On the trading island people with coconuts meet another with probability \( \gamma \).

**Navigation:** Travel to the trading island from any production island is instantaneous. Because of fog, finding a production island is a tricky business. People set out in boats but only hit a random island with probability \( \sigma \) each period. After arriving at an island the individuals have to decide whether to look for coconuts there or keep looking for a better island (i.e. one with shorter trees.)

**Endowments:** Everyone has a boat and starts off with a coconut

(a) Write down a set of Bellman or “asset value” equations that are implied by the above environment. (Hint: you will need 3 equations one each for: looking for trading partner, looking for a suitable island, looking for a coconut on the chosen island)

(b) Let \( c^* \) represent the critical tree height such that no one stops at an island where the trees are taller than \( c^* \). Express \( c^* \) in terms of the value to being on the trading island and the value to looking for a suitable island. How should the value of \( c^* \) change with \( \alpha \), the proportion of trees with coconuts on any island? (If you do not have time to do the algebra make a guess and provide intuition for that guess.)

(c) Draw a diagram showing the flows between the states an individual can be in. Write down the steady state equations that could be used to figure out how many people are in each state.